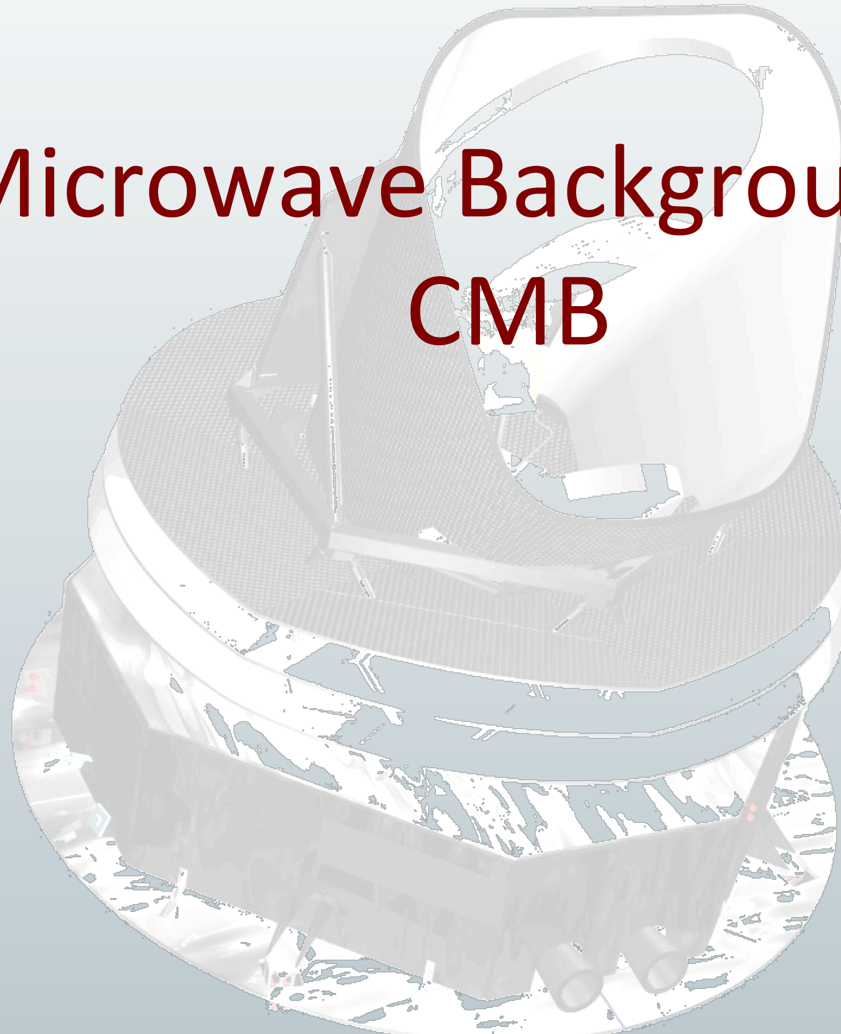
Cosmic Microwave Background Radiation
CMB



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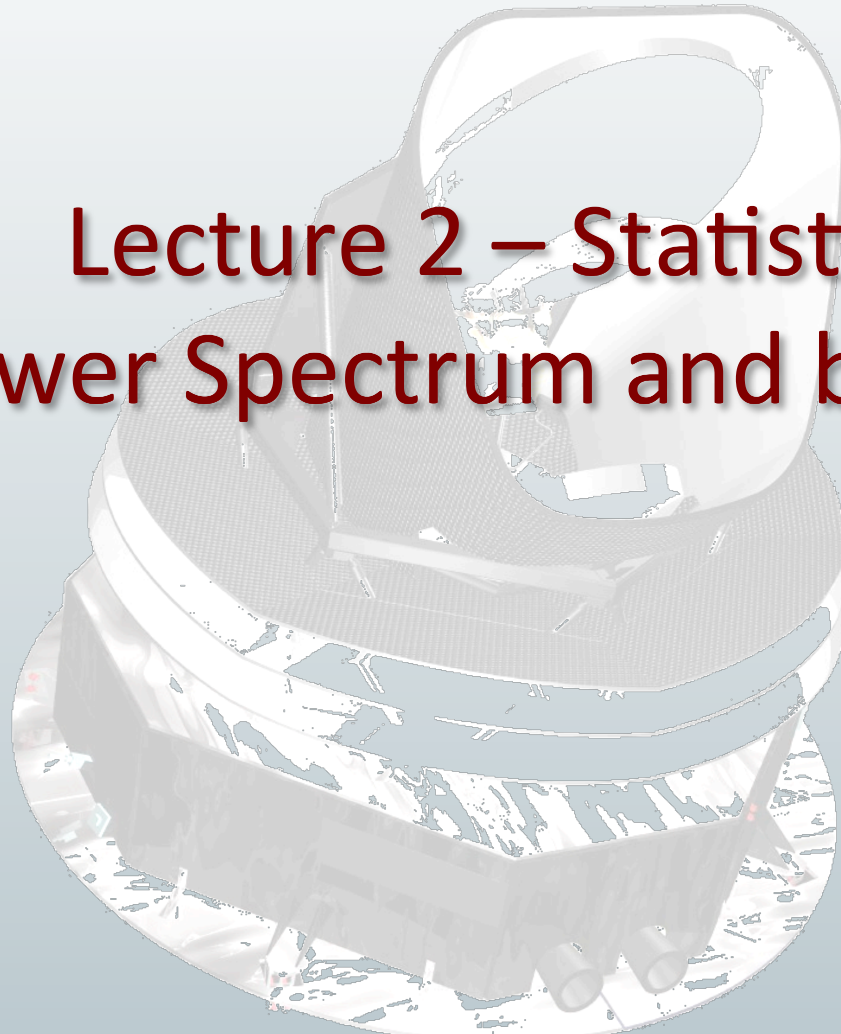
Vth INPE Advance Course on Astrophysics, INPE, September 2013



Lecture 2 – Statistics
Power Spectrum and beyond

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Statistics of Random Fields

- Theory of generation of perturbations is quantum mechanical (eg quantum mechanics during inflation) → limited to predicting the statistical properties of the fluctuations (such as the matter overdensities δρ)

- Mean values (denoted e.g. $\langle \delta\rho \rangle$) represent quantum expectation values
  - Inflation automatically provides 'decoherence', i.e. produces essentially classical fluctuations → can interpret expectation values as over a classical ensemble of universes

- Will consider random fields in three-dimensional Euclidean space (i.e. restrict to flat (K=0) universes) and in two dimensions on the sphere
  - To keep the Fourier analysis simple
  - 'Fourier' analysis more complicated in non-flat models
  - We denote comoving spatial positions by x

- Consider real random scalar field $f(\boldsymbol{x})$ in flat space with mean zero:
  $\langle f(\boldsymbol{x}) \rangle = 0$ ie at each point f(x) is some random number with zero mean
  - Probability of getting $f(\boldsymbol{x})$ is *functional* $\mathrm{Pr}[f(\boldsymbol{x})]$
- Convenient to expand in eigenfunctions of $\boldsymbol{\nabla}^2$ to make translational properties manifest:

$$f(\boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}} f(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \quad ; \quad f(\boldsymbol{k}) = \int \frac{d^3\boldsymbol{x}}{(2\pi)^{3/2}} f(\boldsymbol{x}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}$$

  - Consider (actively) translating field by $\boldsymbol{a}$: $f(\boldsymbol{x}) \to \hat{T}_{\boldsymbol{a}} f = f(\boldsymbol{x} - \boldsymbol{a})$ so that

$$\hat{T}_{\boldsymbol{a}} f(\boldsymbol{k}) = \int \frac{d^3\boldsymbol{x}}{(2\pi)^{3/2}} f(\boldsymbol{x} - \boldsymbol{a}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} = f(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{a}}$$

- Could also rotate field about origin with $\hat{R}$: $f(\boldsymbol{x}) \to \hat{R} f(\boldsymbol{x}) = f(\mathbf{R}^{-1}\boldsymbol{x})$ so that

$$\hat{R} f(\boldsymbol{k}) = \int \frac{d^3\boldsymbol{x}}{(2\pi)^{3/2}} f(\mathbf{R}^{-1}\boldsymbol{x}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} = f(\mathbf{R}^{-1}\boldsymbol{k})$$

- Demand that statistical properties of fluctuations respect the FRW symmetries of homogeneity and isotropy

The Power Spectrum

Path Integral over all field configurations

$$\xi(\mathbf{x},\mathbf{y}) \equiv \langle f(\mathbf{x}) f(\mathbf{y}) \rangle = \int \mathcal{D}f \, \mathrm{Pr}[f] f(\mathbf{x}) f(\mathbf{y}) \, ,$$

- Consider two-point statistics in Fourier space
  - If ensemble is statistically homogeneous, must have $\longrightarrow$ $\mathrm{Pr}[f(x)] = \mathrm{Pr}[\hat{T}_a f(x)]$

$$\langle \hat{T}_{\boldsymbol{a}} f(\boldsymbol{k}) [\hat{T}_{\boldsymbol{a}} f(\boldsymbol{k}')]^* \rangle = \langle f(\boldsymbol{k}) f^*(\boldsymbol{k}') \rangle e^{-i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{a}} = \langle f(\boldsymbol{k}) f^*(\boldsymbol{k}') \rangle \quad \forall \boldsymbol{a}$$

so $\langle f(\boldsymbol{k}) f^*(\boldsymbol{k}') \rangle = F(\boldsymbol{k}) \delta_D(\boldsymbol{k} - \boldsymbol{k}')$ $\longrightarrow$ *different Fourier modes are uncorrelated*

  - If ensemble also statistically isotropic $\longrightarrow$ $\mathrm{Pr}[f(x)] = \mathrm{Pr}[\hat{R} f(x)]$

$$\langle \hat{R} f(\boldsymbol{k}) [\hat{R} f(\boldsymbol{k}')]^* \rangle = \langle f(\mathbf{R}^{-1}\boldsymbol{k}) f^*(\mathbf{R}^{-1}\boldsymbol{k}') \rangle = F(\mathbf{R}^{-1}\boldsymbol{k}) \delta_D(\boldsymbol{k} - \boldsymbol{k}')$$

must equal $F(\boldsymbol{k})\delta_D(\boldsymbol{k} - \boldsymbol{k}')$ for all $\hat{R}$, i.e. $F(\boldsymbol{k}) = F(k)$ where $k = |\boldsymbol{k}|$

$$\delta_D(R^{-1}k) = \det R \delta(k) = \delta(k)$$

- Define power spectrum of homogeneous, isotropic process by

$$\langle f(\boldsymbol{k}) f^*(\boldsymbol{k}') \rangle = \frac{2\pi^2}{k^3} \mathcal{P}_f(k) \delta_D(\boldsymbol{k} - \boldsymbol{k}')$$

(prefactor is convenient, but many different conventions exist)

THE POWER SPECTRUM

Path Integral over all field configurations

$$\xi(\mathbf{x},\mathbf{y}) \equiv \langle f(\mathbf{x}) f(\mathbf{y})\rangle = \int \mathcal{D}f\, \Pr[f]\, f(\mathbf{x}) f(\mathbf{y}),$$

- Consider two-point statistics in Fourier space
  - If ensemble is statistically homogeneous, must have

$$\xi(\mathbf{x},\mathbf{y}) = \xi(\mathbf{x}-\mathbf{a},\mathbf{y}-\mathbf{a}) \quad \forall \mathbf{a}$$
$$\xi(\mathbf{x},\mathbf{y}) = \xi(\mathbf{x}-\mathbf{y}),$$

$$\rangle e^{-i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{a}} = \langle f(\boldsymbol{k}) f^*(\boldsymbol{k}')\rangle \quad \forall \boldsymbol{a}$$

- 2-point correlator only depends on the separation of the 2 points

  - If ensemble also statistically isotropic

- 2-point correlator depends only on the distance between the 2 points

$$\xi(\mathbf{x},\mathbf{y}) = \xi\left(\mathsf{R}^{-1}(\mathbf{x}-\mathbf{y})\right) \quad \forall \mathsf{R}$$
$$\Rightarrow \quad \xi(\mathbf{x},\mathbf{y}) = \xi(|\mathbf{x}-\mathbf{y}|),$$

$$\mathsf{R}^{-1}\boldsymbol{k}'))\rangle = F(\mathsf{R}^{-1}\boldsymbol{k})\delta_D(\boldsymbol{k}-\boldsymbol{k}')$$

must equal $F(\boldsymbol{k})\delta_D(\boldsymbol{k}-\boldsymbol{k}')$ for all R, i.e. $F(\boldsymbol{k}) = F(k)$ where $k = |\boldsymbol{k}|$

- Define power spectrum of homogeneous, isotropic process by

$$\langle f(\boldsymbol{k}) f^*(\boldsymbol{k}')\rangle = \frac{2\pi^2}{k^3}\mathcal{P}_f(k)\delta_D(\boldsymbol{k}-\boldsymbol{k}')$$

(prefactor is convenient, but many different conventions exist)

Two-point correlation functions

- Auto-correlation function of $f(\boldsymbol{x})$ in real space is the two-point correlation function

$$\xi_f(\boldsymbol{x}, \boldsymbol{x}') \equiv \langle f(\boldsymbol{x}) f(\boldsymbol{x}') \rangle$$

  - Since $\langle f(\boldsymbol{x}) \rangle = 0$, ξ_f measures excess over (Poisson) case where f independent at every point

- Inserting Fourier expansions gives

$$\xi_f(\boldsymbol{x}, \boldsymbol{x}') = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{2\pi^2}{k^3} \mathcal{P}_f(k) e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} = \int \frac{dk}{k} \mathcal{P}_f(k) \mathrm{sinc}(k|\boldsymbol{x}-\boldsymbol{x}'|)$$

  - Fourier transform of power spectrum
  - Only depends on $|\boldsymbol{x} - \boldsymbol{x}'|$ as required by statistical homogeneity and isotropy

The variance of the field is $\xi(0) = \int d\ln k P_f(k)$
Scale invariant means P (k)=const., and its variance receives equal contribution from every decade in k

Two-point correlation functions

- Auto-correlation function of $f(\boldsymbol{x})$ in real space is the two-point correlation function

$$\xi_f(\boldsymbol{x},\boldsymbol{x}') = \langle f(\boldsymbol{x})\, f(\boldsymbol{x}')\rangle$$

$$k.(x-y) = k|x-y|\mu$$

$$\langle f(\mathbf{x}) f(\mathbf{y})\rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{d^3\mathbf{k}'}{(2\pi)^{3/2}} \underbrace{\langle f(\mathbf{k}) f^*(\mathbf{k}')\rangle}_{\frac{2\pi^2}{k^3}\mathcal{P}_f(k)\delta(\mathbf{k}-\mathbf{k}')} e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}'\cdot\mathbf{y}}$$

$$= \frac{1}{4\pi}\int \frac{dk}{k}\mathcal{P}_f(k)\int d\Omega_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}.$$

er (Poisson) case where f

$$2\pi\int_{-1}^{1} d\mu\, e^{ik|\mathbf{x}-\mathbf{y}|\mu} = 4\pi j_0(k|\mathbf{x}-\mathbf{y}|),$$

$$\xi_f(\boldsymbol{x},\boldsymbol{x}') = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{2\pi^2}{k^3}\mathcal{P}_f(k) e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} = \int \frac{dk}{k}\mathcal{P}_f(k)\,\mathrm{sinc}(k|\boldsymbol{x}-\boldsymbol{x}'|)$$

  - Fourier transform of power spectrum
  - Only depends on $|\boldsymbol{x}-\boldsymbol{x}'|$ as required by statistical homogeneity and isotropy

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Gaussian processes I

- For a Gausssian process, $\mathrm{Pr}[f(\boldsymbol{x})]$ is a Gaussian functional of $f(\boldsymbol{x})$
- Since $f(\boldsymbol{k})$ is linear in $f(\boldsymbol{x})$, $\mathrm{Pr}[f(\boldsymbol{k})]$ also Gaussian
- Simplifies discussion if consider periodic fields in box volume V: $\boldsymbol{k}$ takes discrete values with lattice volume $(2\pi)^3/V$ so if $f(\boldsymbol{x}) = \sum_{\boldsymbol{k}} f_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$

$$\int d^3\boldsymbol{k} \mapsto \frac{(2\pi)^3}{V}\sum_{\boldsymbol{k}} \quad ; \quad \delta_D(\boldsymbol{k}-\boldsymbol{k}') \mapsto \frac{V}{(2\pi)^3}\delta_{\boldsymbol{k}\boldsymbol{k}'} \quad ; \quad f(\boldsymbol{k}) \mapsto \frac{V}{(2\pi)^{3/2}} f_{\boldsymbol{k}}$$

  - Follows that $\langle f_{\boldsymbol{k}} f^*_{\boldsymbol{k}'}\rangle = \frac{(2\pi)^3}{V}\frac{\mathcal{P}_f(k)}{4\pi k^3}\delta_{\boldsymbol{k}\boldsymbol{k}'}$
- For a homogeneous, isotropic Gaussian process, real and imaginary parts, $a_{\boldsymbol{k}}$ and $b_{\boldsymbol{k}}$, of $f_{\boldsymbol{k}}$ are Gaussian independent variables:

$$\mathrm{Pr}(\{a_{\boldsymbol{k}}, b_{\boldsymbol{k}}\}) = \prod_{\boldsymbol{k}} \frac{e^{-(a^2_{\boldsymbol{k}}+b^2_{\boldsymbol{k}})/2\mu^2(k)}}{2\pi\mu^2(k)} \quad \text{with} \quad 2\mu^2(k) = \frac{(2\pi)^3}{V}\frac{\mathcal{P}_f(k)}{4\pi k^3}$$

  - Reality of $f(\boldsymbol{x}) \rightarrow f(\boldsymbol{k}) = f^*(-\boldsymbol{k})$ so only half variables (linearly) independent

- Changing variables to $r_{\boldsymbol{k}}$ and $\phi_{\boldsymbol{k}}$ s.t. $f_{\boldsymbol{k}} \equiv r_{\boldsymbol{k}} e^{i\phi_{\boldsymbol{k}}}$,

$$\Pr(r_{\boldsymbol{k}}, \phi_{\boldsymbol{k}}) = \frac{1}{2\pi} \frac{r_{\boldsymbol{k}}}{\mu^2(k)} e^{-r_{\boldsymbol{k}}^2/2\mu^2(k)}$$

so amplitudes and phases independent with $\phi_{\boldsymbol{k}}$ uniform on $[0, 2\pi]$

- Returning to probability functional for $f(\boldsymbol{x})$ have

$$\ln \Pr[f(\boldsymbol{x})] = -\frac{1}{2} \sum_{\boldsymbol{k}} \frac{|f_{\boldsymbol{k}}|^2}{2\mu^2(k)} + \text{constant}$$

  - Using $|f_{\boldsymbol{k}}|^2 = V^{-2} \int d^3\boldsymbol{x} d^3\boldsymbol{y}\, f(\boldsymbol{x}) f(\boldsymbol{y}) e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}$, taking $V \to \infty$ find

$$\ln \Pr[f(\boldsymbol{x})] = -\frac{1}{2} \int d^3\boldsymbol{x} d^3\boldsymbol{y}\, f(\boldsymbol{x}) \xi_f^{-1}(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{y}) + \text{constant}$$

  - Inverse correlation function s.t.

$\int d^3\,\boldsymbol{y} \xi_f(\boldsymbol{x} - \boldsymbol{y}) \xi_f^{-1}(\boldsymbol{y} - \boldsymbol{x}') = \delta_D(\boldsymbol{x} - \boldsymbol{x}')$:

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$$\xi_f^{-1}(\boldsymbol{x}, \boldsymbol{y}) \equiv \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \left(\frac{2\pi^2 \mathcal{P}_f(k)}{k^3} \right)^{-1} e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}$$

BEYOND THE POWER SPECTRUM

- Three-point function in Fourier space is *bispectrum*
  - Statistical homogeneity demands

$$\langle f(\boldsymbol{k}_1) f(\boldsymbol{k}_2) f(\boldsymbol{k}_3)\rangle = \delta_D(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3) F(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$$

    with $F(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ permutation symmetric

  - Statistical isotropy and parity invariance ($\boldsymbol{x} \to -\boldsymbol{x}$) demands that F depends only on lengths of sides:

$$\langle f(\boldsymbol{k}_1) f(\boldsymbol{k}_2) f(\boldsymbol{k}_3)\rangle = \delta_D(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3) \mathcal{B}_f(k_1, k_2, k_3)$$

- For (zero-mean) Gaussian processes bispectrum (and all odd n-point functions) vanishes

- More generally $2n$-point functions of (zero-mean) Gaussian fields satisfy Wicks theorem: e.g.

Sum of means of all possible pairs

$$\langle f(\boldsymbol{k}_1) f(\boldsymbol{k}_2) f(\boldsymbol{k}_3) f(\boldsymbol{k}_4)\rangle = \langle f(\boldsymbol{k}_1) f(\boldsymbol{k}_2)\rangle\langle f(\boldsymbol{k}_3) f(\boldsymbol{k}_4)\rangle + \langle f(\boldsymbol{k}_1) f(\boldsymbol{k}_3)\rangle\langle f(\boldsymbol{k}_2) f(\boldsymbol{k}_4)\rangle + \langle f(\boldsymbol{k}_1) f(\boldsymbol{k}_4)\rangle\langle f(\boldsymbol{k}_2) f(\boldsymbol{k}_3)\rangle$$

i.e. sum of means of all possible pairs

Random fields on the sphere

- For a random field $f(\hat{\boldsymbol{n}})$ on the sphere expand in *spherical harmonics* $Y_{lm}(\hat{\boldsymbol{n}})$

$$f(\hat{\boldsymbol{n}}) = \sum_{lm} f_{lm} Y_{lm}(\hat{\boldsymbol{n}})$$

  - Eigenfunctions of spherical Laplacian ∇^2 and ∂_ϕ: (with $\hbar = 1$)

$$\hat{L}^2 Y_{lm} = -\nabla^2 Y_{lm} = l(l+1) Y_{lm}$$

$$\hat{L}_z Y_{lm} = -i\partial_\phi Y_{lm} = m Y_{lm}$$

  with l integer ≥ 0 and m integer with $|m| \leq l$

  - Orthonormal over sphere:

$$\int d\hat{\boldsymbol{n}}\, Y_{lm}(\hat{\boldsymbol{n}}) Y^*_{(lm)'}(\hat{\boldsymbol{n}}) = \delta_{ll'}\delta_{mm'}$$

  - Use phase convention s.t. $Y^*_{lm} = (-1)^m Y_{l\,-m}$ so that $f^*_{lm} = (-1)^m f_{l\,-m}$ for real fields

$$f_{lm} = \int d\hat{n} f(\hat{n}) Y^*_{lm}(\hat{n})$$

Spherical multipole coefficients

- Rotation $\hat{D}$ fully specified by three Euler angles, $\{\alpha, \beta, \gamma\}$:
  - Rotate by γ about z then β about y then α about z
- Since ϕ-dependence of Y_{lm} is $e^{im\phi}$

$$\hat{D}(0,0,\gamma)Y_{lm} = e^{-im\gamma}Y_{lm} = e^{-i\gamma\hat{L}_z}Y_{lm}$$

  - $\hat{L}_i$ is generator of rotations about i-axis, so

$$\hat{D}(\alpha,\beta,\gamma)Y_{lm} = e^{-i\alpha\hat{L}_z}e^{-i\beta\hat{L}_y}e^{-i\gamma\hat{L}_z}Y_{lm}$$

- Since $\hat{L}^2$ commutes with $\hat{L}_i$

$$\hat{D}Y_{lm} = \sum_{m'} D^l_{m'm} Y_{lm'}$$

  - Extract $D^l_{m'm}$ using orthonormality:

$$D^l_{m'm} = \int d\hat{\boldsymbol{n}}\, Y^*_{lm'} e^{-i\alpha\hat{L}_z}e^{-i\beta\hat{L}_y}e^{-i\gamma\hat{L}_z}Y_{lm} = e^{-im'\alpha}\underbrace{\int d\hat{\boldsymbol{n}}\, Y^*_{lm'}e^{-i\beta\hat{L}_y}Y_{lm}}_{d^l_{m'm}(\beta)}\, e^{-im\gamma}$$

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- Consider two point statistic in multipole space: $\langle f_{lm} f^*_{(lm)'} \rangle$
  - Demand invariance under rotations of $f(\hat{\boldsymbol{n}})$ for which

$$f_{lm} \mapsto \hat{D} f_{lm} \equiv \sum_{m'} f_{lm'} \int d\hat{\boldsymbol{n}}\, Y^*_{lm} \hat{D} Y_{lm'} = \sum_{m'} D^l_{mm'} f_{lm'}$$

- Key result we require follows from unitarity of $\hat{D}$:

$$\hat{D}^\dagger \hat{D} = 1 \quad \Rightarrow \quad \int d\hat{\boldsymbol{n}}\, [\hat{D} Y_{lm}]^* \hat{D} Y_{lm'} = \delta_{mm'} \quad \Rightarrow \quad \sum_M D^{l*}_{Mm} D^l_{Mm'} = \delta_{mm'}$$

- Follows that if

$$\langle f_{lm} f^*_{(lm)'} \rangle = \sum_{MM'} D^l_{mM} D^{l'*}_{m'M'} \langle f_{lM} f^*_{(lM)'} \rangle$$

must have

$$\langle f_{lm} f^*_{(lm)'} \rangle = \delta_{ll'} \delta_{mm'} C_l$$

  - Defines spherical power spectrum C_l

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THE SPHERICAL POWER SPECTRUM

- Consider two point statistic in multipole space: $\langle f_{lm} f^*_{(lm)'} \rangle$
  - Demand invariance under rotations of $f(\hat{\boldsymbol{n}})$ for which

$$f_{lm} \mapsto \hat{D} f_{lm} \equiv \sum_{m'} f_{lm'} \int d\hat{\boldsymbol{n}}\, Y^*_{lm} \hat{D} Y_{lm'} = \sum_{m'} D^l_{mm'} f_{lm'}$$

- Key result we require follows from unitarity of $\hat{D}$:

$$\hat{D}^\dagger \hat{D} = 1 \quad \Rightarrow \quad \int d\hat{\boldsymbol{n}}\, [\hat{D} Y_{lm}]^* \hat{D} Y_{lm'} = \delta_{mm'} \quad \Rightarrow \quad \sum_M D^{l*}_{Mm} D^l_{Mm'} = \delta_{mm'}$$

- Follows that if

For rotation through γ about the z-axis,

$$Y_{lm}(\theta,\phi) \to Y_{lm}(\theta, \phi-\gamma) = e^{-im\gamma} Y_{lm}(\theta,\phi) \qquad \Rightarrow \qquad f_{lm} \to e^{-im\gamma} f_{lm}\,.$$

Under rotations,

$$\langle f_{lm} f^*_{l'm'} \rangle \to e^{-im\gamma} e^{im'\gamma} \langle f_{lm} f^*_{l'm'} \rangle\,,$$

so invariance requires the correlator be $\propto \delta_{mm'}$.

must have

$$\langle f_{lm} f^*_{(lm)'} \rangle = \delta_{ll'} \delta_{mm'} C_l$$

- Defines spherical power spectrum C_l

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- Two-point auto-correlation function on sphere is *angular correlation function*

$$\langle f(\hat{\boldsymbol{n}})f(\hat{\boldsymbol{n}}')\rangle = \sum_{(lm)(lm')} \langle f_{lm} f^*_{(lm')}\rangle Y_{lm}(\hat{\boldsymbol{n}}) Y^*_{(lm)'}(\hat{\boldsymbol{n}}') = \sum_l C_l \sum_m Y_{lm}(\hat{\boldsymbol{n}}) Y^*_{lm}(\hat{\boldsymbol{n}}')$$

  – Sum over m simplifies with *addition theorem* (rotationally-invariant so can evaluate with $\hat{\boldsymbol{n}}$ at north pole):

$$\sum_m Y_{lm}(\hat{\boldsymbol{n}}) Y^*_{lm}(\hat{\boldsymbol{n}}') = \frac{2l+1}{4\pi} P_l(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}')$$

  – Correlation function depends only on angle ψ between $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}}'$ as required:

$$\xi_f(\psi) \equiv \langle f(\hat{\boldsymbol{n}})f(\hat{\boldsymbol{n}}')\rangle = \sum_l \frac{2l+1}{4\pi} C_l P_l(\cos\psi)$$

  – Can invert using orthogonality of Legendre polynomials:

$$C_l = 2\pi \int d\cos\psi\, \xi_f(\psi) P_l(\cos\psi)$$

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- Mean square of field $\xi_f(0) = \sum_l (2l+1)C_l/4\pi$ has power per decade $\sim l^2 C_l/2\pi$

$\sim l^2 C_l/2\pi$

- Consider real field $f(\boldsymbol{x}, \eta)$ in a flat universe
- Observer at $\boldsymbol{x} = 0, \eta = \eta_0$ integrates f along the line of sight $\hat{\boldsymbol{n}}$ to obtain projection

$$f(\hat{\boldsymbol{n}}) \equiv \int d\chi\, f(\chi\hat{\boldsymbol{n}}, \eta_0 - \chi)$$

  - Fourier expansion and Rayleigh $e^{i\chi\boldsymbol{k}\cdot\hat{\boldsymbol{n}}} = 4\pi \sum_{lm} i^l j_l(k\chi) Y_{lm}(\hat{\boldsymbol{n}}) Y^*_{lm}(\hat{\boldsymbol{k}})$ give

$$f_{lm} = \int d\hat{n} f(\hat{n}) Y^*_{lm}(\hat{n})$$

Orthogonality of Y_{lm}

$$f_{lm} = 4\pi i^l \int d\chi \int \frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}} f(\boldsymbol{k}, \eta_0 - \chi) j_l(k\chi) Y^*_{lm}(\hat{\boldsymbol{k}})$$

  - Since $\langle f(\boldsymbol{k}, \eta) f(\boldsymbol{k}', \eta')\rangle = (2\pi^2/k^3)\mathcal{P}_f(k; \eta, \eta')\delta_D(\boldsymbol{k} - \boldsymbol{k}')$, have

$$\langle f_{lm} f^*_{(lm)'}\rangle = \delta_{ll'}\delta_{mm'} \underbrace{4\pi \int d\chi \int d\chi' \int \frac{dk}{k} \mathcal{P}_f(k; \eta_0 - \chi, \eta_0 - \chi') j_l(k\chi) j_l(k\chi'}_{C_l^f}$$

  - Statistical homogeneity and isotropy of $f(\boldsymbol{x}, \eta)$ ensures $f(\hat{\boldsymbol{n}})$ statistically isotropic

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Projections along the line of sight II: Limber approximation

- Spherical Bessel functions $j_l(k\chi)$ arise from intersection of plane wave with sphere of comoving radius χ
  - For large l, $j_l(k\chi)$ peaks sharply at $k\chi \sim l$
  - For smooth $\mathcal{P}_f(k;\eta,\eta')/k^3$, can approximate

$$\int \frac{dk}{k}\mathcal{P}_f(k;\eta,\eta')j_l(k\chi)j_l(k\chi') \approx \left.\frac{\mathcal{P}_f(k;\eta,\eta')}{k^3}\right|_{k\chi=l} \underbrace{\int dk\, k^2 j_l(k\chi)j_l(k\chi')}_{\frac{\pi}{2\chi^2}\delta_D(\chi-\chi')}$$

- Power spectrum C_l^f Limber approximates to

$$C_l^f \approx \frac{2\pi^2}{l^3}\int d\chi\, \chi \mathcal{P}_f(l/\chi;\eta_0-\chi,\eta_0-\chi)$$

  - Equivalent to assuming mapping between l and k so sharp that get no correlations between contributions to projection from different redshifts

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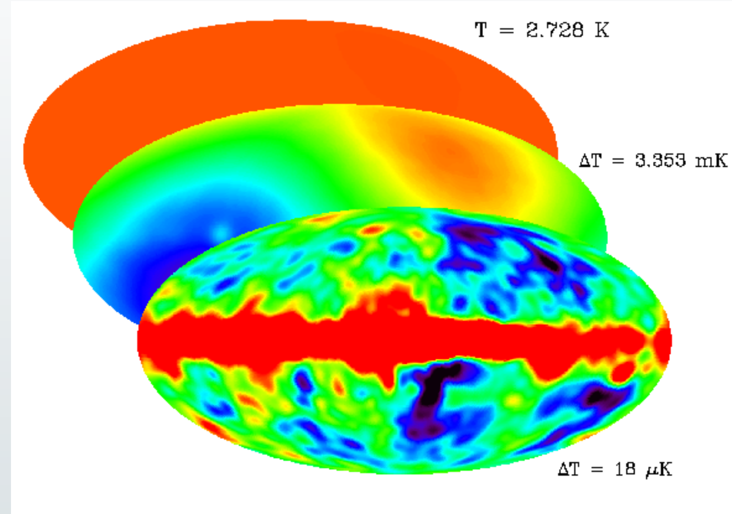
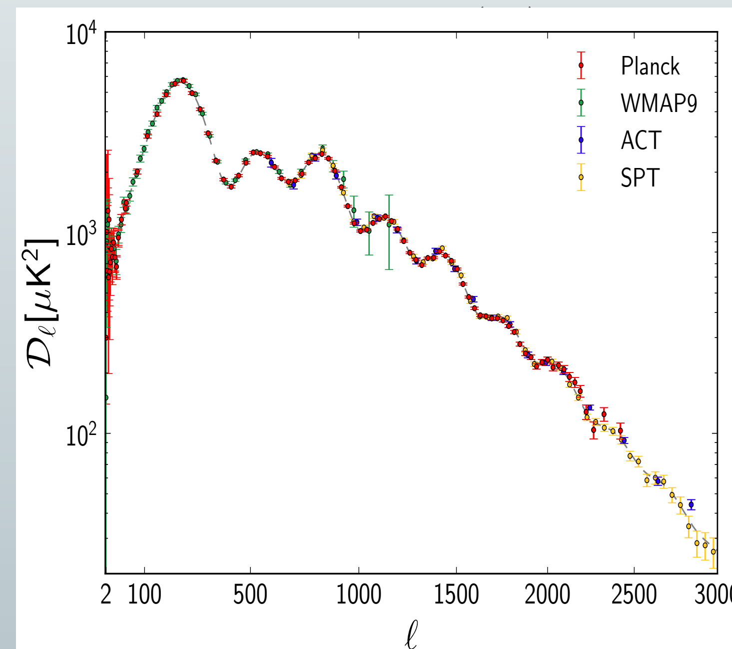
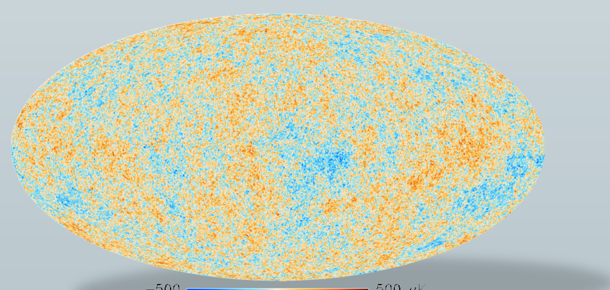
Physics of the CMB

- Microwave background almost perfect blackbody radiation
  - Temperature (COBE-FIRAS) 2.728 K
- Dipole anisotropy $\Delta T/T = \beta \cos\theta$ implies solar-system barycenter has velocity $v/c \equiv \beta = 0.00123$ relative to 'rest-frame' of CMB
- Variance of intrinsic fluctuations first detected by COBE-DMR: $(\Delta T/T)_{\rm rms} = 16\mu{\rm K}$ smoothed on 7° scale
- Anisotropy now detected up to l of few thousand







1

- Work in conformal Newtonian gauge in this lecture and next:

ψ and ϕ are scalar functions of η and x^i;

$$ds^2 = a^2(\eta)[(1+2\psi)d\eta^2 - (1-2\phi)\gamma_{ij}dx^i dx^j]$$

- Using conformal time $d\eta \equiv dt/a$ and comoving coordinates x^i
- γ_{ij} metric for space of constant curvature; in spherical coordinates χ, θ, ϕ:

$$\gamma_{ij}dx^i dx^j = d\chi^2 + \sin_K^2 \chi (d\theta^2 + \sin^2\theta d\phi^2)$$

- Introduce frame of vectors:

$$(E_0)^\mu = a^{-1}(1-\psi)\delta_0^\mu \quad , \quad (E_i)^\mu = a^{-1}(1+\phi)\delta_i^\mu$$

defined so that $(E_0)^a(E_0)_a = 1$ and $(E_i)^a(E_j)_a = -\gamma_{ij}$

- Decompose photon momentum into energy [seen by observer with $u^a = (E_0)^a$] E and direction e^a ($e^a u_a = 0$):

$$p^a = E(u^a + e^a)$$

  - Decompose e^a onto spatial triad $\{(E_i)^a\}$: $e^a = e^i(E_i)^a$

$$e^a e_a = -1 \quad \Rightarrow \quad \gamma_{ij}e^i e^j = 1$$

  - Photon 4-momentum then parameterised in transparent way:

$$p^\mu = a^{-1}E[1-\psi, (1+\phi)e^i]$$

GEODESIC MOTION

- Photons move on geodesics of perturbed spacetime:

$$dp^\mu/d\lambda + \Gamma^\mu_{\nu\rho} p^\nu p^\rho = 0$$

where λ is *affine parameter* with $dx^a/d\lambda = p^a$

- With $p^\mu = a^{-1} E[(1-\psi), (1+\phi)e^i]$ find

(overdot $\equiv \partial/\partial_\eta$):

$$d(aE)/d\eta = -aE d\psi/d\eta + aE(\dot{\psi} + \dot{\phi})$$

$$de^i/d\eta + e^j e^{k\,(3)}\Gamma^i_{jk} = -(\gamma^{ij} - e^i e^j) D_j(\psi - \phi) - (\phi + \psi) e^j e^{k\,(3)}\Gamma^i_{jk}$$

where D_i is covariant derivative from γ_{ij}

  – In background have $aE = \mathrm{const.} \rightarrow$ usual redshift modified by variation of gravitational potentials
  – Derivative $d\psi/d\eta$ is along path of photon
  – Departure of $de^i/d\eta + e^j e^{k\,(3)}\Gamma^i_{jk}$ from zero describes *gravitational lensing*

Temperature anisotropies I

- Integrate $d(aE)/d\eta$ from point A on last scattering surface to observation point R:

$$[\ln aE]_A^R = -[\psi]_A^R + \int_A^R (\dot{\psi} + \dot{\phi})\, d\eta$$

- Total redshift of photon from A to E is $1 + z \equiv E_A/E_R$:

$$1 + z = \frac{a_R}{a_A} \exp\left([\psi]_A^R - \int_A^R (\dot{\psi} + \dot{\phi})\, d\eta \right) \approx \frac{a_R}{a_A} \left(1 + [\psi]_A^R - \int_A^R (\dot{\psi} + \dot{\phi})\, d\eta \right)$$

- Assume radiation isotropic with temperature T_A at A in frame moving with coordinate velocity V_γ^i $\rightarrow$ Doppler shifting to zero-shear frame gives required temperature at A:

$$T_A(1 + e_i V_\gamma^i)$$

- CMB temperature at R in direction e^i is then

$$T_R(e^i) = T_A(1 + e_i V_\gamma^i)_A/(1 + z)$$

TEMPERATURE ANISOTROPIES II

- CMB temperature at R in direction e^i evaluates to

$$T_R(e^i) = \frac{a(\eta_A)}{a(\eta_R)} T_A \left(1 + e_i V_\gamma^i|_A - [\psi]_A^R + \int_A^R (\dot{\psi} + \dot{\phi})\, d\eta \right)$$

- Simplify by noting $T^4 \propto \rho_\gamma$ in frame where isotropic:
  - Since ρ frame-invariant in linear theory have

$$T_A = \bar{T}_A(1 + \tfrac{1}{4}\delta_\gamma)$$

  - But $a_A \bar{T}_A / a_R = \bar{T}_R$ so *gauge-dependent* temperature fluctuation $\Theta(\hat{n}^i) \equiv [T_R(-\hat{n}^i) - \bar{T}_R]/\bar{T}_R$ becomes

$$\Theta(\hat{n}^i) = \frac{1}{4}\delta_\gamma|_A + e_i V_\gamma^i|_A + \psi|_A + \int_A^R (\dot{\psi} + \dot{\phi})\, d\eta$$

  - Have dropped monopole term ψ_R since only contributes to gauge-dependent part of temperature fluctuation (angular variation is gauge-invariant)

ANISOTROPY POWER SPECTRUM

- For *flat universe* extraction of the multipoles a_{lm} of $\Theta(\hat{\boldsymbol{n}})$ follows example in Lecture 32
  - Fourier expand fluctuations and note $\boldsymbol{k} \cdot \boldsymbol{x}_A = \hat{\boldsymbol{n}} \cdot \boldsymbol{k}(\eta_0 - \eta_*)$ with η_* last scattering
  - For Doppler term need (with $\Delta\eta \equiv \eta_0 - \eta_*$)

$$e_i V_\gamma^i|_A = -i \int \frac{d^3 \boldsymbol{k}}{(2\pi)^{3/2}} \hat{\boldsymbol{n}} \cdot \boldsymbol{k} V_\gamma(\boldsymbol{k}) e^{i\hat{\boldsymbol{n}}\cdot\boldsymbol{k}\Delta\eta} = -\frac{d}{d\Delta\eta} \int \frac{d^3 \boldsymbol{k}}{(2\pi)^{3/2}} V_\gamma(\boldsymbol{k}) e^{i\hat{\boldsymbol{n}}\cdot\boldsymbol{k}\Delta\eta}$$

- Multipoles given by

$$a_{lm} = i^l \frac{4\pi}{2l+1} \int \frac{d^3 \boldsymbol{k}}{(2\pi)^{3/2}} \Theta_l(k) \mathcal{R}(\boldsymbol{k}, 0) Y_{lm}^*(\hat{\boldsymbol{k}})$$

with $$\Theta_l(k)\mathcal{R}(\boldsymbol{k},0) = (2l+1)\{[\tfrac{1}{4}\delta_\gamma(\boldsymbol{k},\eta_*) + \psi(\boldsymbol{k},\eta_*)] j_l(k\Delta\eta) - kV_\gamma(\boldsymbol{k},\eta_*) j_l'(k\Delta\eta) + \int_{\eta_*}^{\eta_0} (\dot{\psi} + \dot{\phi})(\boldsymbol{k},\eta) j_l[k(\eta_0 - \eta)]\}$$

- Power spectrum evaluates to

$$C_l = 4\pi \int d\ln k\, \mathcal{P}_\mathcal{R}(k) \Theta_l^2(k)$$

$$\underbrace{\langle \mathcal{R}(0,\mathbf{k}) \mathcal{R}(0,\mathbf{k}')^* \rangle}_{\frac{2\pi^2}{k^3}\mathcal{P}_\mathcal{R}(k)\delta(\mathbf{k}-\mathbf{k}')}$$

SACHS-WOLFE EFFECT I

- On large scales more convenient to express $\frac{1}{4}\delta_\gamma + \psi$ in terms of $\tilde{\delta}_\gamma$ in *comoving gauge*
  - Under (scalar) gauge transformation $\eta \mapsto \eta + T$ and $x^i \mapsto x^i + D^i L$ total matter velocity V and metric fluctuation B transform to

$$\tilde{V} = V + \dot{L} \quad , \quad \tilde{B} = B + T - \dot{L}$$

  - In comoving gauge $\tilde{V} = 0$ and metric fluctuation $\tilde{B} = 0 \rightarrow$ reach from CNG with

$$T = -V \quad \text{and} \quad \dot{L} = -V$$

  - Density fluctuations transform as $\delta \mapsto \delta - T\dot{\rho}/\rho$ so in comoving gauge

$$\tfrac{1}{4}\tilde{\delta}_\gamma = \tfrac{1}{4}\delta_\gamma - V\mathcal{H}$$

- In CNG, Einstein field equations give $\dot{\phi} + \mathcal{H}\psi = -\frac{1}{2}\kappa a^2(\rho + p)V$ so

$$\frac{1}{4}\delta_\gamma + \psi = \frac{1}{4}\tilde{\delta}_\gamma - \frac{2\mathcal{H}\dot{\phi}}{\kappa a^2(\rho + p)} + \frac{\psi(1 + 3p/\rho)}{3(1 + p/\rho)}$$

Sachs-Wolfe effect II

- If matter dominates radiation at last scattering $\phi \approx \psi$ and $\dot{\phi}/\mathcal{H} \ll \phi$ temperature anisotropy reduces to

$$\Theta(\hat{n}^i) = \tfrac{1}{3}\phi|_A + \tfrac{1}{4}\tilde{\delta}_\gamma|_A + e_i V_\gamma^i|_A + 2\int_A^R \dot{\phi}\, d\eta$$

  - ISW term receives only late-time contribution as Λ becomes dominant

- CNG field equations give $(D^2 + 3K)\phi \approx \tfrac{3}{2}\mathcal{H}^2\tilde{\delta}$
  - For *adiabatic* initial perturbations $\tilde{\delta}_\gamma \sim \tilde{\delta}$ so $\tilde{\delta}_\gamma \sim k^2\phi/\mathcal{H}^2$
  - Show later that on large scales $e_i V_\gamma^i \sim k\phi/\mathcal{H}$

$$\int_0^\infty j_l^2(x)\, dx = \frac{1}{2l(l+1)}$$

- On large scales *Sachs-Wolfe* and ISW dominate

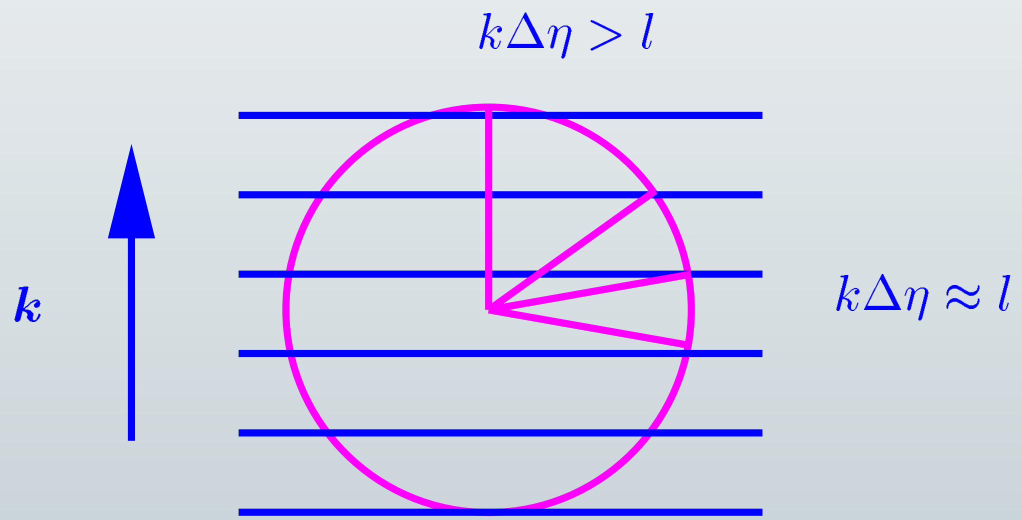
- Power spectrum of Sachs-Wolfe part on large scales for scale-invariant $\mathcal{P}_\mathcal{R}(k) = \mathrm{const.}$:

$$C_l = \frac{4\pi}{9}\left[\frac{\sqrt{\mathcal{P}}\phi(\boldsymbol{k},\eta_*)}{\mathcal{R}(\boldsymbol{k},0)}\right]^2 \int d\ln k\, j_l^2(k\Delta\eta) = \frac{2\pi}{25}\frac{\mathcal{P}}{l(l+1)}$$

where have used conservation of $\mathcal{R}$ on large scales and $\mathcal{R} = 5\phi/3$ in matter domination

Acoustic peaks

- Bessel functions in $\Theta_l(k)$ describe projection between angular scale l and linear scale k at last scattering
  - $j_l(k\Delta\eta)$ peaks when $k\Delta\eta \approx l$ but for given l considerable power from $k > l/\Delta\eta$ also (wavefronts perpendicular to line of sight)



- Acoustic oscillations in the photon-baryon fluid on sub-(sound) horizon scales have k-dependent frequency so phase of oscillation at η_* is k-dependent
  - Expect modulation in C_ls on sub-degree scales where k is inside sound horizon

PHOTON-BARYON DYNAMICS I

- Before recombination Thomson scattering keeps photons isotropic in frame of baryons $V_\gamma = V_b$ and preserves adiabaticity $\delta_\gamma = 4\delta_b/3$
  - Coupled photon-baryon system behaves almost like ideal fluid: (where $R \equiv 3\rho_b/4\rho_\gamma$)

$$\delta\rho = \rho_\gamma\delta_\gamma + \rho_b\delta_b = \rho_\gamma\delta_\gamma(1+R) \quad , \quad \delta p = \tfrac{1}{3}\rho_\gamma\delta_\gamma$$

- CNG continuity and momentum equations for ideal fluid:

$$\partial_\eta\delta\rho + (\rho+p)D^2V + 3[(\delta\rho+\delta p)\mathcal{H} - (\rho+p)\dot{\phi}] = 0$$

$$\dot{V} + \mathcal{H}V + \frac{\dot{p}}{\rho+p}V + \psi + \frac{\delta p}{\rho+p} = 0$$

  - Applied to interacting photon-baryon fluid find

$$\dot{\delta}_\gamma + \tfrac{4}{3}D^2V_\gamma - 4\dot{\phi} = 0 \quad , \quad \dot{V}_\gamma + \tfrac{R}{1+R}\mathcal{H}V_\gamma + \tfrac{1}{4(1+R)}\delta_\gamma + \psi = 0$$

- On super-Hubble scales (for adiabatic fluctuations) $\phi \approx \mathrm{const.}$ and $\dot{\delta}_\gamma \sim \mathcal{H}\delta_\gamma$ and $\dot{V}_\gamma \sim \mathcal{H}V_\gamma$ so

$$\mathcal{H}\delta_\gamma \sim \tfrac{4}{3}k^2V_\gamma \Rightarrow \mathcal{H}V_\gamma \sim \psi \quad \text{so} \quad e_iV_\gamma^i \sim kV_\gamma \sim k\psi/\mathcal{H} \ll \psi$$

- Combine continuity and momentum equations to get

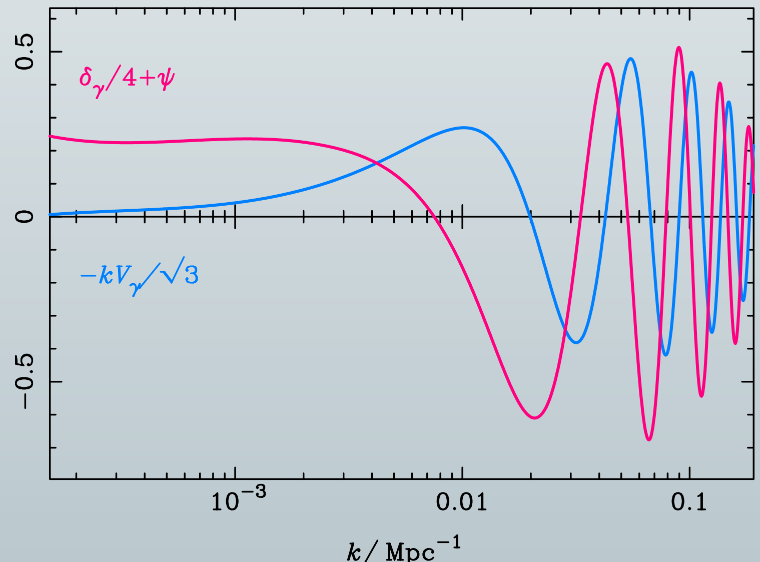
$$\ddot{\delta}_\gamma + \frac{\mathcal{H}R}{1+R}\dot{\delta}_\gamma - \frac{1}{3(1+R)}D^2\delta_\gamma = 4\ddot{\phi} + \frac{4\mathcal{H}R}{1+R}\dot{\phi} + \frac{4}{3}D^2\psi$$

  - Sound speed $c_s^2 = \frac{1}{3}(1+R)^{-1}$ reduced by baryon inertia

- Neglecting baryons ($R \approx 0$) and evolution of potentials have approximate solution

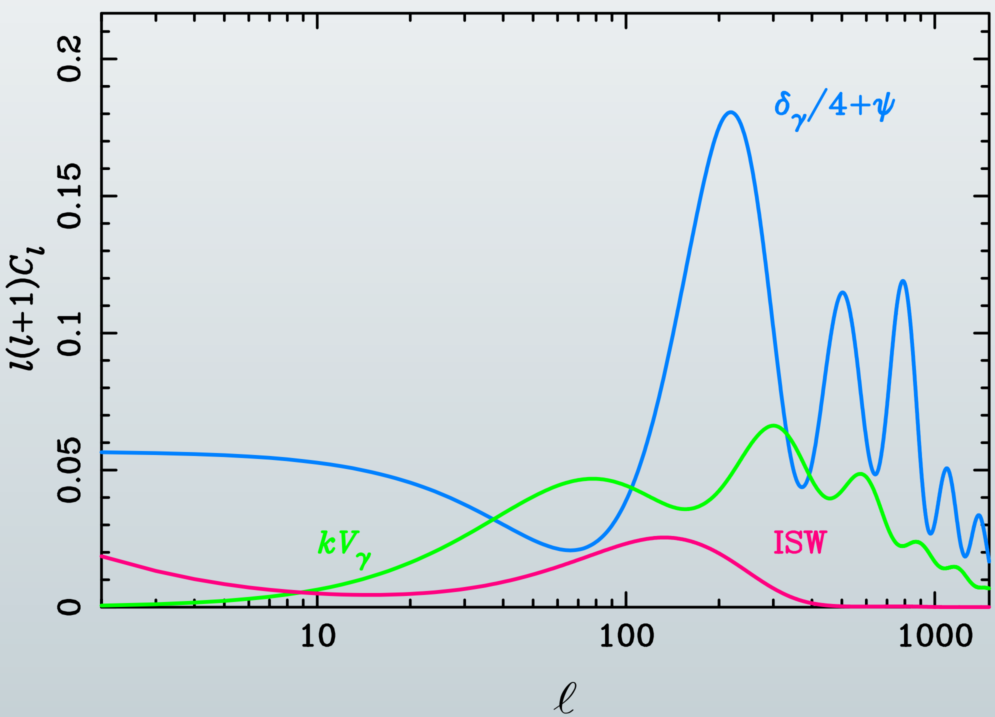
$$\frac{1}{4}\delta_\gamma(\eta) + \psi(\eta) = [\frac{1}{4}\delta_\gamma(0) + \psi(\eta)]\cos(k\eta/\sqrt{3}) + \frac{\sqrt{3}}{k}\dot{\delta}_\gamma(0)\sin(k\eta/\sqrt{3})$$

- Super-Hubble adiabatic fluctuations have $\delta_\gamma \approx -2\psi \Rightarrow \dot{\delta}_\gamma(0) \approx 0$
- Photon velocity follows from continuity equation
- Extrema of cosine oscillation when $k\int_0^{\eta_*} c_s\, d\eta' = n\pi$

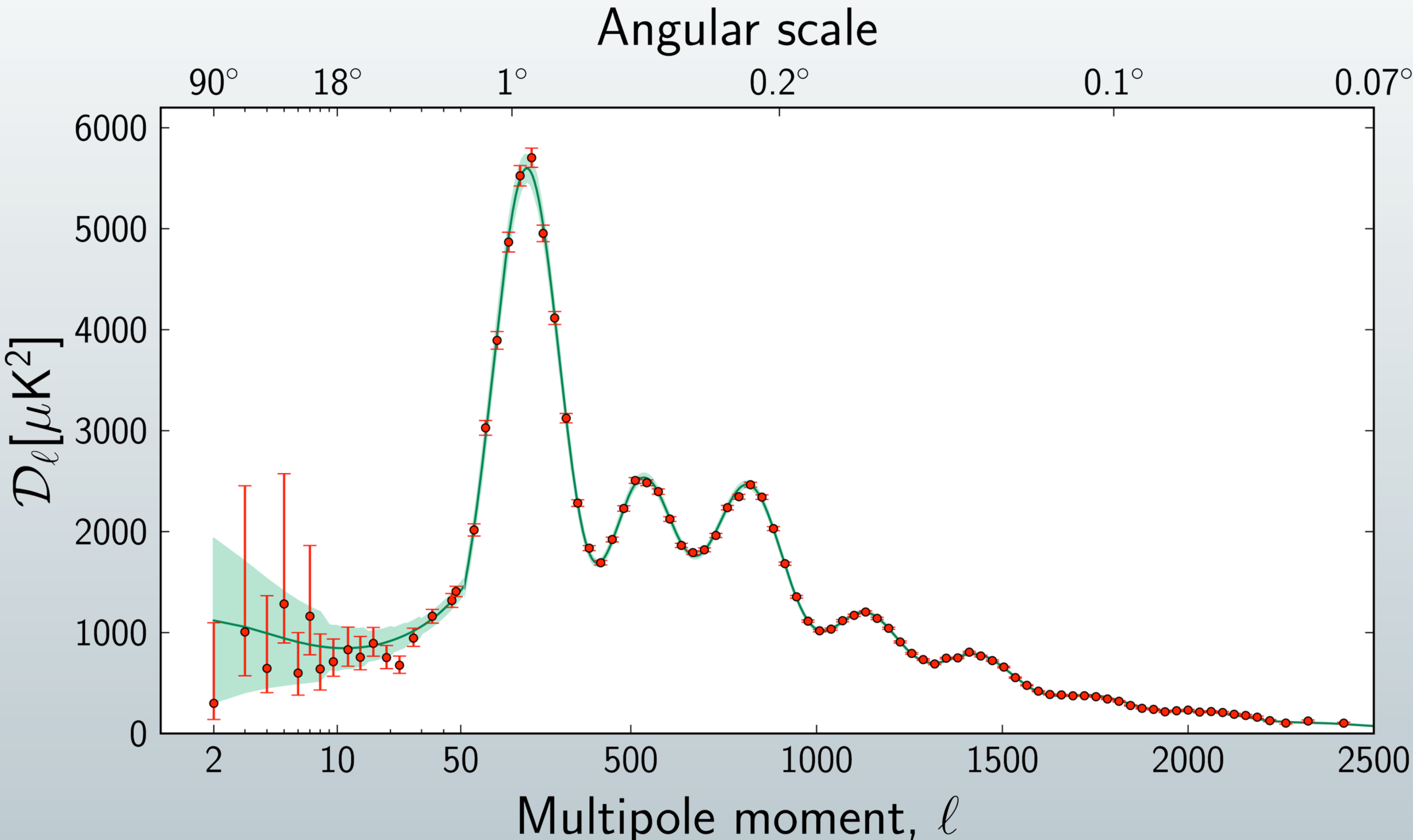


SUMMARY

- Redshift effects (Sachs-Wolfe and ISW) dominate on large scales
- Intrinsic temperature fluctuation dominant source of acoustic peaks, but minima filled in by dipole
  - Early ISW (due to residual radiation density at last scattering) important contribution to first acoustic peak
- Simple photon-baryon fluid model inadequate on small scales (Lec-



Planck Power Spectra (TT)



Appendix

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Vth INPE Advance Course on Astrophysics, INPE, September 2013