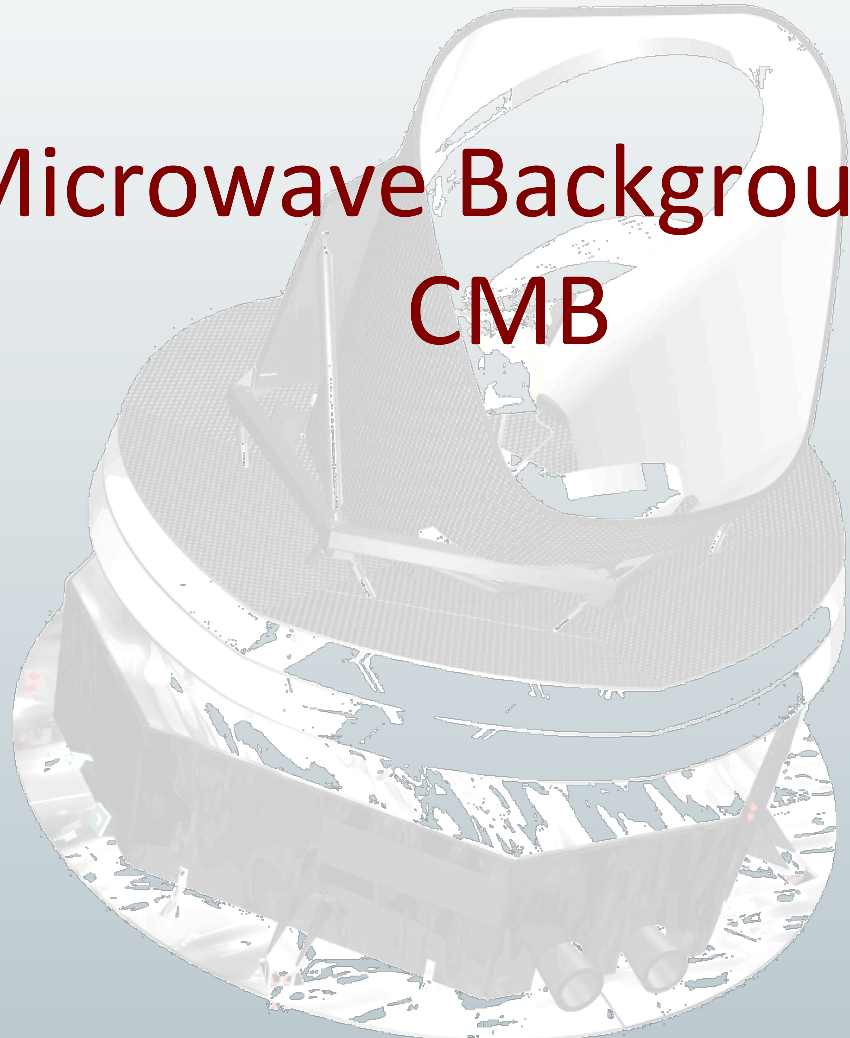
Cosmic Microwave Background Radiation
CMB



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Vth INPE Advance Course on Astrophysics, INPE, September 2013

Lecture 1 – Cosmological Perturbations

L1.1 Newtonian Perturbation Theory

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Cosmological perturbations

- Newtonian gravity adequate on scales well inside Hubble radius where spacetime curvature can be neglected

- Need General relativity for super-Hubble scales and relativistic matter (pressure comparable to total energy density; e.g. radiation)

- Here we will give an account of perturbation theory for sub-Hubble scales ie Newtonian perturbation theory

- Consider a gravitating fluid with mass density ρ, pressure p, and velocity **v:**
  Need to solve a fluid dynamic problem – apply the 3 conservations laws:

  Conservation of mass – continuity equation
  Conservation of momentum – Navier-Stokes equation
  Conservation of energy – Poisson equation

LECTURE 1: NEWTONIAN PERTURBATION THEORY

- Newtonian gravity adequate on scales well inside Hubble radius where spacetime curvature can be neglected
- Need general relativity for super-Hubble scales and relativistic matter (pressure comparable to total energy density; e.g. radiation)
- Consider gravitating fluid with (mass) density ρ, pressure p and velocity $\boldsymbol{v}$; described by continuity, Navier Stokes and Poisson equations:

$$\begin{aligned} \partial_t \rho + \boldsymbol{\nabla}_{\boldsymbol{r}} \cdot (\rho \boldsymbol{v}) &= 0 \\ \partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}} \boldsymbol{v} &= -\frac{1}{\rho} \boldsymbol{\nabla} p - \boldsymbol{\nabla}_{\boldsymbol{r}} \Phi \\ \boldsymbol{\nabla}^2_{\boldsymbol{r}} \Phi &= \frac{1}{2} \kappa \rho \end{aligned}$$

with $\kappa \equiv 8\pi G$

- Can fudge up cosmological constant by $\boldsymbol{\nabla}^2_{\boldsymbol{r}} \Phi = \frac{1}{2} \kappa \rho - \Lambda$

Background dynamics

- Consider uniform density with isotropic expansion $\Longrightarrow \quad \boldsymbol{v} = H(t)\boldsymbol{r}$ (Hubble)
- With $\Phi = 0$ at origin, Poisson equation integrates to

$$\Phi = \tfrac{1}{12} r^2(\kappa\rho - 2\Lambda)$$

- Navier Stokes equation determines evolution of H:

$$\partial_t H + H^2 = -\tfrac{1}{6}\kappa\rho + \tfrac{1}{3}\Lambda$$

- Continuity equation gives mass dilution by expansion:

$$\partial_t \rho + 3H\rho = 0 \quad \Longrightarrow \quad \rho \propto a^{-3} \quad \text{with} \quad \partial_t a = Ha$$

- Equations have first integral $a^2(H^2 - \kappa\rho/3) - \Lambda a^2/3$; follows that

$$H^2 + \frac{K}{a^2} = \tfrac{1}{3}\kappa\rho + \tfrac{1}{3}\Lambda$$

with K a constant (GR relates K/a^2 to spatial curvature)

Perturbation analysis I: comoving coordinates

- Comoving observer in background has

$$\boldsymbol{v} = \partial_t \boldsymbol{r} = H(t)\boldsymbol{r}$$

- Use coordinates fixed w.r.t. background comoving observers: $\boldsymbol{x} = \boldsymbol{r}/a(t)$
- Not same as Lagrangian coordinates (fixed w.r.t. real perturbed fluid)
- Derivatives transform as

$$\begin{aligned} \left(\frac{\partial}{\partial t}\right)_{\boldsymbol{r}} &= \left(\frac{\partial}{\partial t}\right)_{\boldsymbol{x}} + \left(\frac{\partial \boldsymbol{x}}{\partial t}\right)_{\boldsymbol{r}} \cdot \boldsymbol{\nabla} = \left(\frac{\partial}{\partial t}\right)_{\boldsymbol{x}} - H(t)\boldsymbol{x} \cdot \boldsymbol{\nabla} \\ \boldsymbol{\nabla}_{\boldsymbol{r}} &= \frac{1}{a}\boldsymbol{\nabla} \end{aligned}$$

with $\boldsymbol{\nabla} \equiv \boldsymbol{\nabla}_{\boldsymbol{x}}$

Perturbation analysis II: dynamics

- Perturb around background solution, i.e.

$$\rho \to \rho + \delta\rho, \quad \boldsymbol{v} \to Ha\boldsymbol{x} + \delta\boldsymbol{v}, \quad \Phi \to \Phi + \delta\Phi$$

with ρ and Φ background values

- Perturbed dynamics in comoving coordinates:

$$\begin{aligned}
\partial_t\delta\rho + a^{-1}\rho\boldsymbol{\nabla}\cdot\delta\boldsymbol{v} + a^{-1}\delta\boldsymbol{v}\cdot\boldsymbol{\nabla}\delta\rho + 3H\delta\rho + a^{-1}\delta\rho\boldsymbol{\nabla}\cdot\delta\boldsymbol{v} &= 0 \\
\partial_t\delta\boldsymbol{v} + H\delta\boldsymbol{v} + a^{-1}\delta\boldsymbol{v}\cdot\boldsymbol{\nabla}\delta\boldsymbol{v} + a^{-1}(\rho+\delta\rho)^{-1}\boldsymbol{\nabla}\delta p + a^{-1}\boldsymbol{\nabla}\delta\Phi &= 0 \\
\boldsymbol{\nabla}^2\delta\Phi - \tfrac{1}{2}a^2\kappa\delta\rho &= 0
\end{aligned}$$

- Linear theory accurate on large scales (drop products of small quantities):

$$\begin{aligned}
\partial_t\delta + a^{-1}\boldsymbol{\nabla}\cdot\delta\boldsymbol{v} &= 0 \quad (\delta \equiv \delta\rho/\rho) \\
\partial_t\delta\boldsymbol{v} + H\delta\boldsymbol{v} &= -a^{-1}\rho^{-1}\boldsymbol{\nabla}\delta p - a^{-1}\boldsymbol{\nabla}\delta\Phi \\
\boldsymbol{\nabla}^2\delta\Phi &= \tfrac{1}{2}\kappa a^2\rho\delta
\end{aligned}$$

Scalar/vector decomposition

- Can always decompose $\delta \boldsymbol{v}$ as gradient of scalar and divergence-free vector:

$$\delta \boldsymbol{v} = \boldsymbol{v}_{\parallel} + \boldsymbol{v}_{\perp} = \boldsymbol{\nabla} V + \boldsymbol{v}_{\perp}$$

- In linear theory, scalar and vector modes decouple
- For vector modes $\delta\rho = 0 = \delta\Phi$ and

$$\partial_t \delta \boldsymbol{v}_{\perp} + H \delta \boldsymbol{v}_{\perp} = 0 \quad \Longrightarrow \quad \boldsymbol{v}_{\perp} \propto a^{-1}$$

- Vorticity of fluid $\omega = a^{-1} \boldsymbol{\nabla} \times \boldsymbol{v}_{\perp}$ decays as a^{-2} $\longrightarrow$ circulation theorem
- Vorticity only important if actively sourced (e.g. cosmic strings)

Multiple fluids

- For multiple components (radiation, CDM, baryons etc.) with peculiar velocities $\delta \boldsymbol{v}_i$ and density contrasts δ_i, non-relativistic components evolve as

$$\begin{aligned} \partial_t \delta_i + a^{-1} \boldsymbol{\nabla} \cdot \delta \boldsymbol{v}_i &= 0 \\ \partial_t \delta \boldsymbol{v}_i + H \delta \boldsymbol{v}_i &= -a^{-1} \rho^{-1} \boldsymbol{\nabla} \delta p_i - a^{-1} \boldsymbol{\nabla} \delta \Phi \\ \boldsymbol{\nabla}^2 \delta \Phi &= \tfrac{1}{2} \kappa a^2 \sum_i \rho_i \delta_i \end{aligned}$$

- After matter/radiation equality, total fluid variables (usually) dominated by non-relativistic components:

$$\rho \equiv \sum_i \rho_i \qquad p \equiv \sum_i p_i$$

$$\delta \equiv \sum_i \rho_i \delta_i / \rho \qquad \delta \boldsymbol{v} \equiv \sum_i \rho_i \delta \boldsymbol{v}_i / \rho$$

- Then get Newtonian evolution of total matter variables on sub-Hubble scales

Scalar perturbations I: Jeans' length

- Combine hydrodynamic equations into second-order equation for density contrast of single dominant component:

$$\partial_t^2\delta + 2H\partial_t\delta - \tfrac{1}{2}\kappa\rho\delta - \tfrac{1}{a^2\rho}\boldsymbol{\nabla}^2\delta p = 0$$

- Consider barotropic equation of state $p = p(\rho)$ with sound speed^2 $= v_s^2 = \partial p/\partial\rho$
- Fluctuations with comoving wavenumber k, $\boldsymbol{\nabla}^2\delta = -k^2\delta$:
  - Exponentially unstable for $v_s^2 k^2 < \tfrac{1}{2}\kappa a^2\rho$ if no expansion
  - Reduced to power-law instability in expanding models
- Jeans' (proper) length $\lambda_J = v_s\sqrt{\pi/(G\rho)}$ separates gravitationally stable modes from unstable (stable if sound crossing time less than gravitational free-fall time)

Scalar perturbations II: solutions in EdS

- After matter/radiation equality to late times, universe close to Einstein-de Sitter (i.e. $p \approx 0$, K and Λ negligible for background evolution) $\longrightarrow a \propto t^{2/3}$
- Fluctuations in *dominant* non-relativistic component on sub-Hubble scales

$$\partial_t^2 \delta + \frac{4}{3t}\partial_t \delta - \frac{2}{3t^2}\delta - \frac{v_s^2}{a^2}\boldsymbol{\nabla}^2\delta = 0$$

- Solutions outside Jeans' scale: $\delta \propto t^{2/3}, \quad t^{-1}$
- Growing mode solution $\propto a$: maintains constant $\delta\Phi$ from $\boldsymbol{\nabla}^2\delta\Phi = \frac{1}{2}\kappa a^2 \rho\delta$
- For fluctuations on comoving scale $k \gg k_J$ ($k_J/a = \sqrt{\frac{2}{3}\frac{1}{v_s t}}$), WKB solution is

$$\delta \propto \sqrt{\frac{k_J}{k}} t^{-1/6} \sin\left(\sqrt{\frac{2}{3}} \int \frac{k}{k_J}\frac{dt}{t} + \psi\right)$$

Damped oscillation with adiabatically slow variation of spring and damping constants

CDM evolution in radiation domination

- Consider CDM fluctuations on sub-Hubble scales in radiation domination
- Below its Jeans' scale, radiation perturbations oscillate rapidly cf. expansion rate:

$$\sum_i \langle \rho_i \delta_i \rangle \approx \rho_{DM} \delta_{DM}$$

since baryons tightly-coupled to radiation by Compton scattering

- CDM evolution approximates to

$$\partial_t^2 \delta_{DM} + \frac{1}{t} \partial_t \delta_{DM} - \frac{1}{2} \kappa \rho_{DM} \delta_{DM} \approx 0$$

since $H = 1/(2t)$ in radiation domination

- As $H^2 \gg \kappa \rho_{DM}$, approximate solutions $\delta_{DM} \propto$ constant, $\ln t$
- Slow growth of sub-Hubble CDM fluctuations until matter comes to dominate

Late time suppression of matter perturbation growth

- Consider epochs when dominant *matter* components are pressure-free (i.e. CDM and baryons); total density contrast evolves as:

$$\partial_t^2\delta + 2H\partial_t\delta - \frac{1}{2}\kappa\rho\delta = 0$$

- In open models $K < 0$, with $\Lambda = 0$, curvature could dominate matter at late times:
  - $a \propto t$ and $H^2 \gg \kappa\rho$:

$$\partial_t^2\delta + \frac{2}{t}\partial_t\delta \approx 0$$

  - Solutions $\delta \propto$ constant, $t^{-1} \longrightarrow$ curvature suppression of growth

- In Λ models, cosmological constant dominates at late times:
  - $a \propto \exp(\sqrt{\Lambda/3}t)$ and $H^2 \gg \kappa\rho_{DM}$

$$\partial_t^2\delta + 2\sqrt{\frac{\Lambda}{3}}\partial_t\delta \approx 0$$

  - Solutions $\delta \propto$ constant, $\exp(-2\sqrt{\Lambda/3}t) \longrightarrow$ suppression by Λ

Evolution of baryon fluctuations

- Prior to recombination baryons tightly-coupled to photons
  - Oscillation of photon/baryon fluid on scales below Jeans' scale
  - CDM grows after matter/radiation equality so $|\delta_{DM}| \gg |\delta_B|$
- After recombination (but when Λ and K negligible) can ignore baryon pressure:

$$\begin{aligned} \partial_t^2 \delta_{DM} + \frac{4}{3t}\partial_t \delta_{DM} &\approx \frac{1}{2}\kappa(\rho_B \delta_B + \rho_{DM}\delta_{DM}) \\ \partial_t^2 \delta_B + \frac{4}{3t}\partial_t \delta_B &\approx \frac{1}{2}\kappa(\rho_B \delta_B + \rho_{DM}\delta_{DM}) \end{aligned}$$

- 'Normal coordinates' are δ and $\delta_{DM} - \delta_B$:

$$\delta \approx \frac{\rho_{DM}}{\rho}\delta_{DM} + \frac{\rho_B}{\rho}\delta_B \propto t^{2/3}, \quad t^{-1}; \qquad \delta_{DM} - \delta_B \propto \text{constant}, \quad t^{-1/3}$$

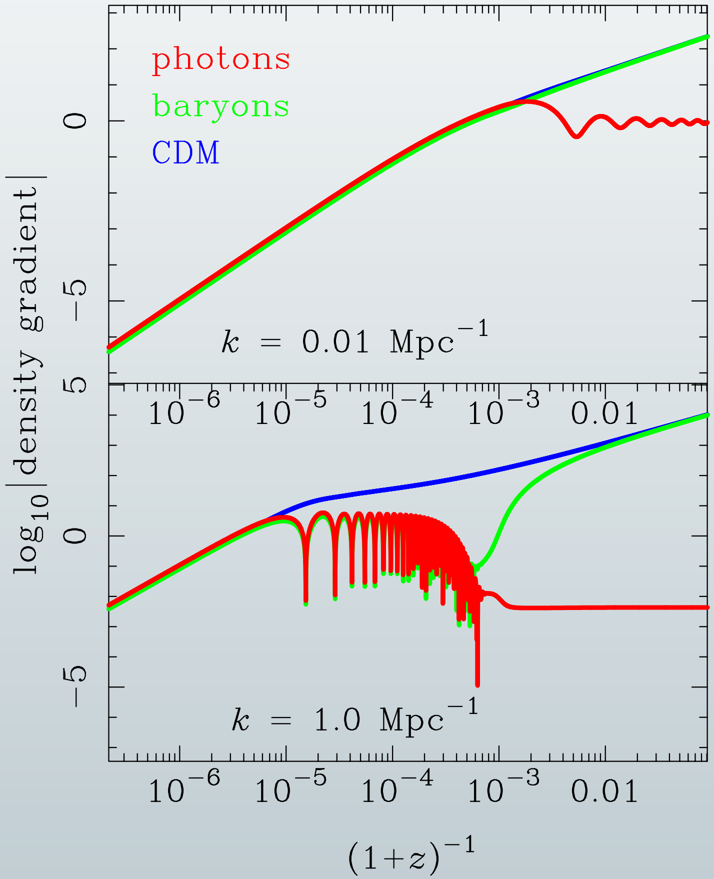
- Asymptotically have $\delta_{DM} = \delta_B \propto t^{2/3}$, i.e. baryons 'catch up' with CDM through gravity

Summary of perturbation evolution

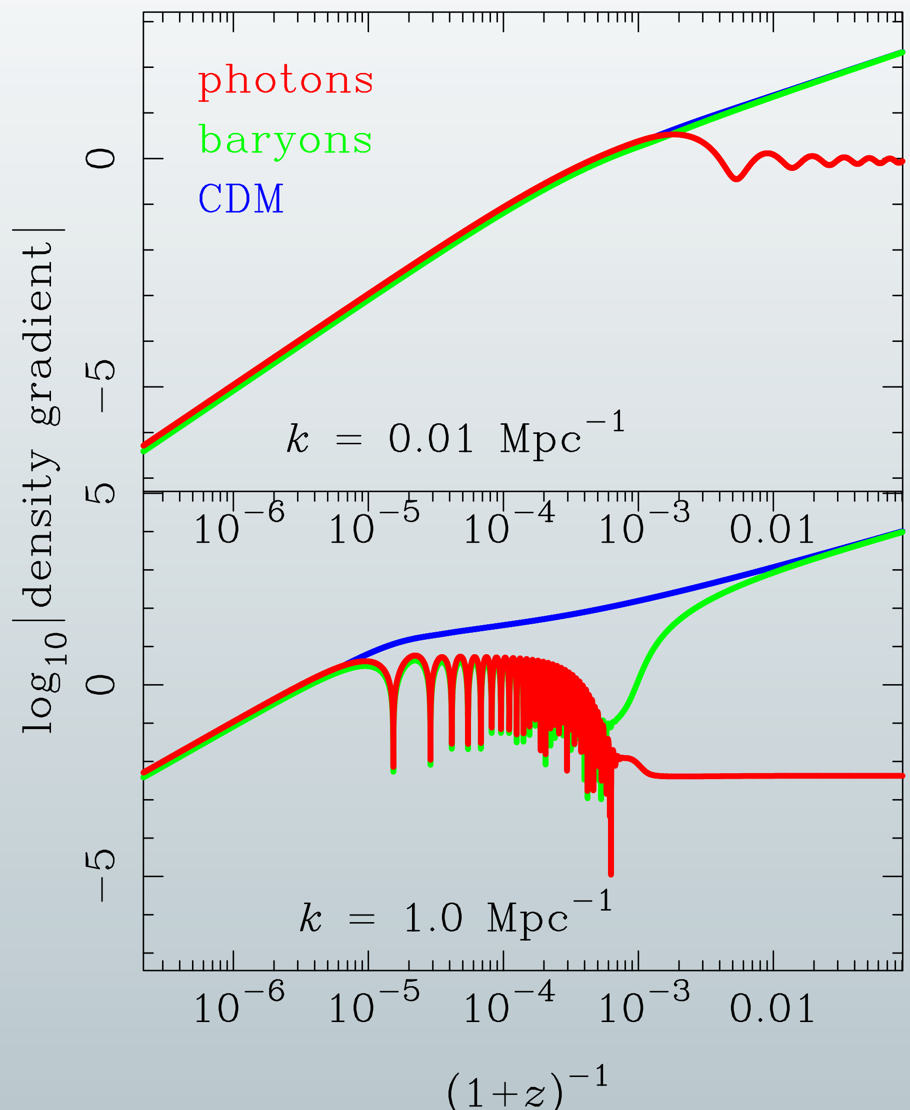
- Evolution of perturbations in ΛCDM model for adiabatic initial conditions:

$$\delta_{DM} = \delta_B = \frac{3}{4}\delta_\gamma$$

- Remain adiabatic when super-Hubble (but δ_i gauge-dependent)
- Sub-Hubble, CDM growth slow until matter dominates, then power law
- Before recombination, tightly-coupled baryons and photons oscillate acoustically inside Jeans' scale
- After recombination baryons fall into CDM potential wells



Summary of perturbation evolution



Lecture 2 – Cosmological Perturbations

L1.2 Relativistic Perturbation Theory

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Relativistic perturbation theory

- The dynamics of perturbations to FRW universe are described by the Einstein's field equations and depend on the nature of the energy-momentum tensor

- To study the evolution of non-relativistic dust:
  - use the perfect fluid model with p=0
  - derive the equations describing linear perturbations in the synchronous gauge from Einstein's equations and the energy-momentum tensor

- To describe the evolution of perturbations in the photons and neutrinos - need to specify the evolution of a distribution function $f(\mathbf{x}, \mathbf{p}, t)$ which specifies the number of particles per unit volume in phase-space:

$$\delta N = f(\mathbf{x}, \mathbf{p}, t)\delta^3\mathbf{x}\delta^3\mathbf{p}$$

such evolution is given by the Boltzmann equation and therefore we need to study the perturbations to the Boltzmann equation in General Relativity, GR

Boltzmann equation

- Energy-momentum tensor is:

$$T^{ij} = \int \frac{d^4p}{(-g)^{1/2}} 2\delta(g^{lm}p_l p_m - m^2) p^i p^j f$$

where δ is the δ-function here and the momenta $p_\alpha = mu_\alpha$ satisfy the geodesic equation $p_0 \frac{dp_i}{dt} = \frac{1}{2} g_{jk,i} p^j p^k$

Define p such that $p_0^2 = p^2 + m^2$ (energy-momentum relation in Minkowski coordinates)

In the presence of a gravitational field $\longrightarrow g^{ij} p_i p_j = m^2$

The collisionless Boltzmann equation is:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^\alpha}\frac{dx^\alpha}{dt} + \frac{\partial f}{\partial p_0}\frac{dp_0}{dt} + \frac{\partial f}{\partial \gamma^\alpha}\frac{d\gamma^\alpha}{dt} = 0$$

where γ_α are the direction cosines of p_α wrt our spatial coordinates. Last term can be dropped (2nd oder in the perturbation variables). Need expression for dx^α/dt and dp_0/dt to first order

26

Boltzmann equation

It can be shown that:

$$\frac{dp_0}{dt} \sim -\frac{p^2}{p_0}\left[-2\frac{\dot{a}}{a} + \dot{h}_{\alpha\beta}\gamma_\alpha\gamma_\beta\right]$$
$$\frac{dx^\alpha}{dt} = -\frac{p_\alpha}{ma^2} \sim \frac{p\gamma_\alpha}{p_0 a}$$

Write the distribution function as

$$f = f_0 + f_1$$

where f_0 is the unperturbed distribution function and f_1 is the perturbation

- The gravitational field equations yield for the evolution of the perturbation to the metric tensor:

$$\frac{\partial^2 h}{\partial t^2} + \frac{2}{a}\frac{da}{dt}\frac{\partial h}{\partial t} = 16\pi G \int d^3p f_1 \left(p_0 - \frac{m^2}{2p_0}\right)$$
$$\frac{\partial^2}{\partial x^\alpha \partial t}(h\delta_{\alpha\beta} - h_{\alpha\beta}) = -16\pi G a \int d^3p p f_1 \gamma_\beta$$

27

Boltzmann equation

where $h = tr(h_{ij})$

All is needed to describe the evolution of a system of collisionless massive particles in linear theory is the collisionless Boltzmann equation, the perturbation to the metric equation and the evolution of the homogeneous background universe.

Define comoving momenta $q_0 = p_0 a$, $q = pa$ and conformal time $\tau = \int \frac{dt}{a}$,
from now on dots will denote differentiation wrt τ

- the perturbation to the distribution function evolves as:

$$\dot{f}_1 + ik\mu \frac{q}{q_0} f_1 = -\frac{1}{2}\frac{\partial f_0}{\partial q} q \left[\dot{h}\frac{(1-\mu^2)}{2} + \dot{h}_{33}\frac{(3\mu^2-1)}{2} \right]$$

after Fourier transforming the equation, with $\mathbf{k}$ along the 3-axis and $\mu = \hat{\mathbf{k}} \cdot \mathbf{q}$,
and $(\frac{\partial f_0}{\partial \tau})_q = 0$

For photons $m = 0$ and so $q = q_0$, defining Δ as the perturbation to the

Boltzmann equation

radiation brightness we have:

$$\Delta(k,\mu,\tau) = f_1\left(\frac{T_0}{4}\frac{\partial f_0}{\partial T_0}\right)^{-1}$$

so, the collisionless (free streaming) Boltzmann equation becomes:

$$\dot{\Delta} + ik\mu\Delta = [\dot{h}(1-\mu^2) + \dot{h}_{33}(3\mu^2-1)] \qquad (3)$$

while the equations governing the time evolution of h_{ij} on the new system of coordinates:

$$\ddot{h} + \frac{\dot{a}}{a}\dot{h} = \frac{8\pi G}{a^2}\int d^3q f_1\left(q_0 + \frac{q^2}{q_0}\right) \qquad (4)$$

$$ik(\dot{h}_{33} - \dot{h}) = \frac{16\pi G}{a^2}\int d^3q q\mu f_1 \qquad (5)$$

Note that for massless particles the evolution of each Fourier mode of Δ depends only on the $\angle(\mathbf{k},\mathbf{q})$ and not on the energy of the particles (differs from massive particles case)

Perturbations I

- Consider only baryonic matter and photons (can be generalized to include massless and massive neutrinos) $\longrightarrow$ describe the baryonic matter as an ideal fluid and the photons by its distribution function $\longrightarrow$ Equations 5 become:

$$\begin{aligned} \ddot{h} + \frac{\dot{a}}{a}\dot{h} &= 8\pi G a^2(\bar{\rho}_m\delta_m + 2\bar{\rho}_\gamma\delta_\gamma) \\ ik(\dot{h}_{33} - \dot{h}) &= 16\pi G(\bar{\rho}_m v + \bar{\rho}_\gamma f_\gamma) \end{aligned}$$

where $\bar{\rho}_m, \bar{\rho}_\gamma \longrightarrow$ matter and photon densities in the background while δ_m and v are the baryon density and velocity perturbations , and $\delta_\gamma = \frac{1}{2}\int_{-1}^{1}\Delta d\mu$, $f_\gamma = \frac{1}{2}\int_{-1}^{1}\delta\mu d\mu$.

The Boltzmann equation for the photons including Thomson scattering is:

$$\dot{\Delta} + ik\mu\Delta + \Phi = \sigma_T n_e a[\delta_\gamma + 4\mu v - \Delta]$$

where it is assumed that Thomson scattering is isotropic (neglect polarization of

30

radiation, and angular dependence of Thomsosn scattering to natural radiation)

- Φ describes the effect of the gravitational field on photons:

$$\Phi = -(3\mu^2 - 1)\dot{h}_{33} - (1 - \mu^2)\dot{h}$$

- the equation of motion of matter in the synchronous gauge:

$$\dot{v} + \frac{v}{a}\dot{a} = \sigma_T n_e a \frac{\bar{\rho}_\gamma}{\bar{\rho}_m}\left(f_\gamma - \frac{4}{3}v\right) \tag{6}$$

- baryonic density perturbation obeys:

$$\dot{\delta}_m = \frac{1}{2}\dot{h} - ikv$$

Consider the evolution of optically thick perturbations $\longrightarrow$ in the tight coupling

31

Perturbations I

limit $kt_c \ll 1$, $t_c = 1/(\sigma_T n_e) \ll t$:

$$\Delta \sim \delta_\gamma + 4\mu v - \frac{1}{\sigma_T n_e} a[\dot{\delta}_\gamma + 4\mu\dot{v} + ik\mu(\delta_\gamma + 4\mu v) + \Phi + \cdots] \qquad (7)$$

Zeroth moment:

$$\dot{\delta}_\gamma = \frac{4}{3}\dot{\delta}_m$$

defining an entropy perturbation $\delta S_\gamma = (\frac{3}{4}\delta_\gamma - \delta_m)$, which is gauge invariant $\longrightarrow$ if the initial plane-wave perturbation is adiabatic $(\delta S_\gamma = 0)$ then it will remain so in the optically thick limit (assume that fluctuations are adiabatic when $k\tau \ll 1$)

The first angular momentum of equation 7 gives photon energy flux:

$$f_\gamma = \frac{4}{3}v - \frac{1}{\sigma_T n_e a}\left[\frac{4}{3}\dot{v} + \frac{1}{3}ik\delta_\gamma\right]$$

Perturbations I

Inserting this equation into 6:

$$\left[\bar{\rho}_m + \frac{4}{3}\bar{\rho}_\gamma\right]\dot{v} + \bar{\rho}_m v \frac{\dot{a}}{a} = -\frac{ik}{3}\delta_\gamma \bar{\rho}_\gamma$$

ie matter and radiation act like a tightly coupled fluid with density $\bar{\rho}_m + \bar{\rho}_\gamma$ and pressure $p = \frac{1}{3}\bar{\rho}_\gamma$

33

Perturbations II- Radiation dominated era

- scale factor $a \propto t^{1/2}$
- neglecting $\bar{\rho}_m$ Equation 6 becomes:

$$\frac{d^2 h}{d^2 t} + 2\frac{\dot{a}}{a}\frac{dh}{dt} \sim 16\pi G \bar{\rho}_\gamma \delta_\gamma$$

For perturbations much larger then the Hubble radius (ie $k\tau \ll 1$) $\longrightarrow$ the effect of pressure gradients can be ignored ie $ik\mu\Delta$ is small compared to $\dot{\Delta}$ so zeroth moment of 7 is $\frac{d\delta_\gamma}{dt} \sim \frac{2}{3}\frac{dh}{dt}$ so,

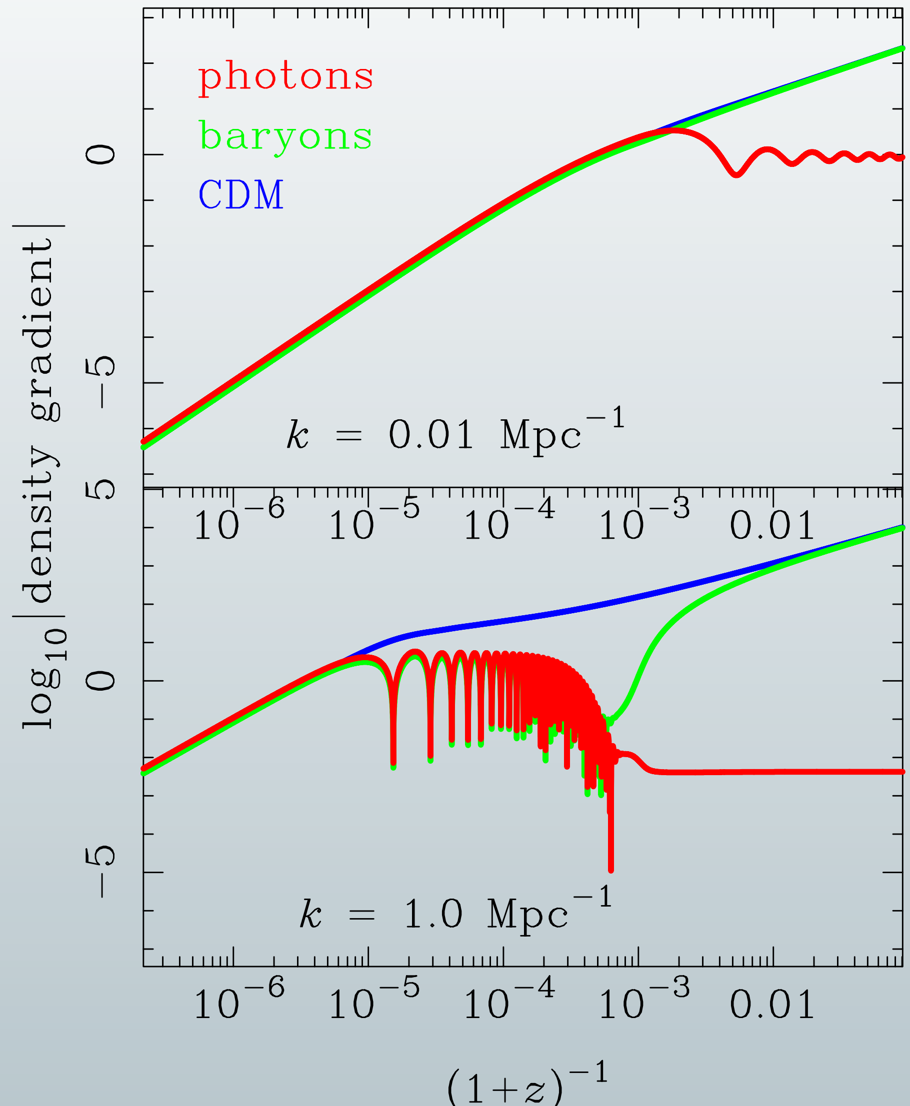
$$\frac{d^2 \delta_\gamma}{dt^2} + \frac{1}{t}\frac{d\delta_\gamma}{dt} = \frac{1}{t^2}\delta_\gamma$$

with solutions:

$$\delta_\gamma = At + Bt^{-1}$$

The decaying mode has no physical significance

Summary of perturbation evolution



Appendix

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