Fundamentals of Radio Interferometry

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Fourth INPE Advanced Course on Astrophysics:
Radio Astronomy in the 21st Century

Topics

- Intensity, Luminosity, Flux, Flux Density, etc.
- The Role of the 'Sensor' (a.k.a. 'Antenna')
- Key Properties of Antennas
- The Need for Interferometry
- The Basic Interferometer

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Spectral Flux Density and Brightness

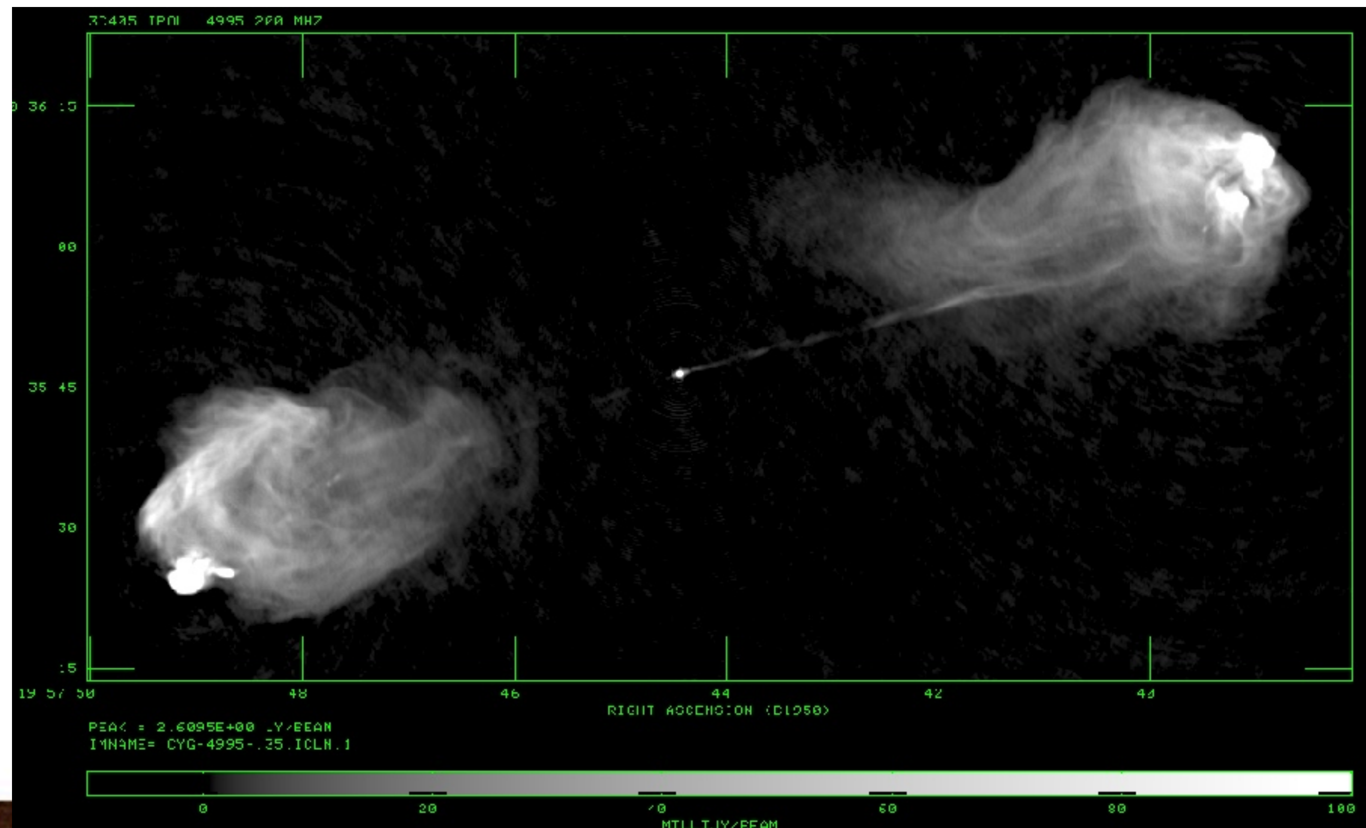
- **Our Goal**: To measure the characteristics of celestial emission from a given direction $\mathbf{s}$, at a given frequency ν, at a given time t.
- In other words: We want a map, or image, of the emission.
- **Terminology/Definitions**: The quantity we seek is called the brightness (or specific intensity): It is denoted here by $I(\mathbf{s},\nu,t)$, and expressed in units of: watt/(m^2 Hz ster).
- It is the power received, per unit solid angle from direction **s**, per unit collecting area, per unit frequency at frequency ν.
- Do not confuse I with Flux Density, S -- the integral of the brightness over a given solid angle:

$$S = \int I(\mathbf{s},\upsilon,t)\,d\Omega$$

- The units of S are: watt/(m^2 Hz)
- Note: 1 Jy = 10^{-26} watt/(m^2 Hz).

An Example – Cygnus A

- I show below an image of Cygnus A at a frequency of 4995 MHz.
- The units of the brightness are Jy/beam, with 1 beam = 0.16 arcsec^2
- The peak is 2.6 Jy/beam, which equates to 6.5 x 10^{-15} watt/(m^2 Hz ster)
- The flux density of the source is 370 Jy = 3.7 x 10^{-24} watt/(m^2 Hz)



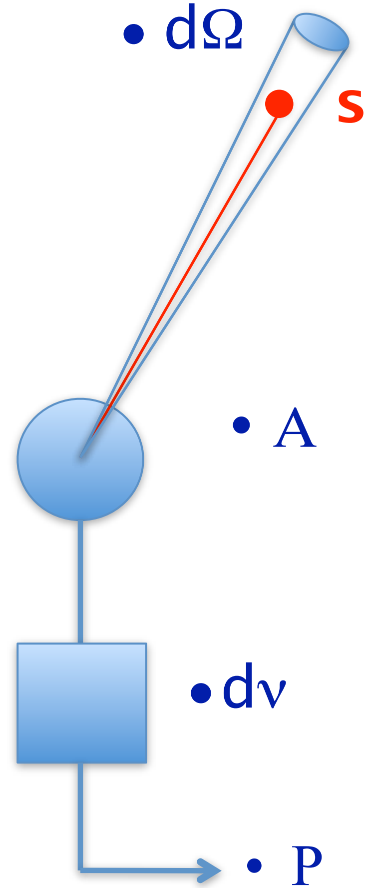
Intensity and Power.

- Imagine a distant source of emission, described by brightness $I(\nu,\mathbf{s})$ where **s** is a unit direction vector.
- Power from this emission is intercepted by a collector ('sensor') of area $A(\nu,\mathbf{s})$.
- The power, P (watts) from a small solid angle $d\Omega$, within a small frequency window $d\nu$, is

$$P = I(\nu, \mathbf{s}) A(\nu, \mathbf{s}) d\nu d\Omega$$

- The total power received is an integral over frequency and angle, accounting for variations in the responses.

$$P = \iint I(\upsilon,\mathbf{s}) A(\upsilon,\mathbf{s}) d\upsilon\, d\Omega$$



- $d\Omega$ Solid Angle
- A Sensor Area
- $d\nu$ Filter width
- P Power collected

The Role of the Sensor

- Coherent interferometry is based on the ability to correlate the electric fields measured at spatially separated locations.
- To do this (without mirrors) requires conversion of the electric field $E(\mathbf{r},\nu,t)$ at some place to a voltage $V(\nu,t)$ which can be conveyed to a central location for processing.
- For our purpose, the sensor (a.k.a. 'antenna') is simply a device which senses the electric field at some place and converts this to a voltage which faithfully retains the amplitudes and phases of the electric fields.
- One can imagine two kinds of ideal sensors:
  - An 'all-sky' sensor: All incoming electric fields, from all directions, are uniformly summed.
  - The 'limited-field-of-view' sensor: Only the fields from a given direction and solid angle (field of view) are collected and conveyed.
- Sadly – neither of these is possible.

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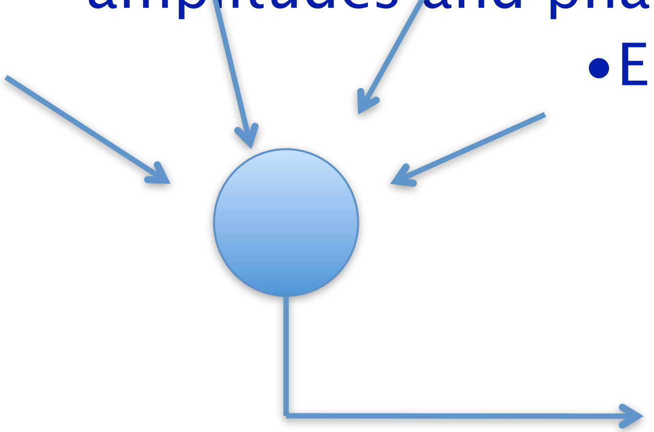


- EM waves in
- Voltage out (preserving amplitude and phase of all input fields)

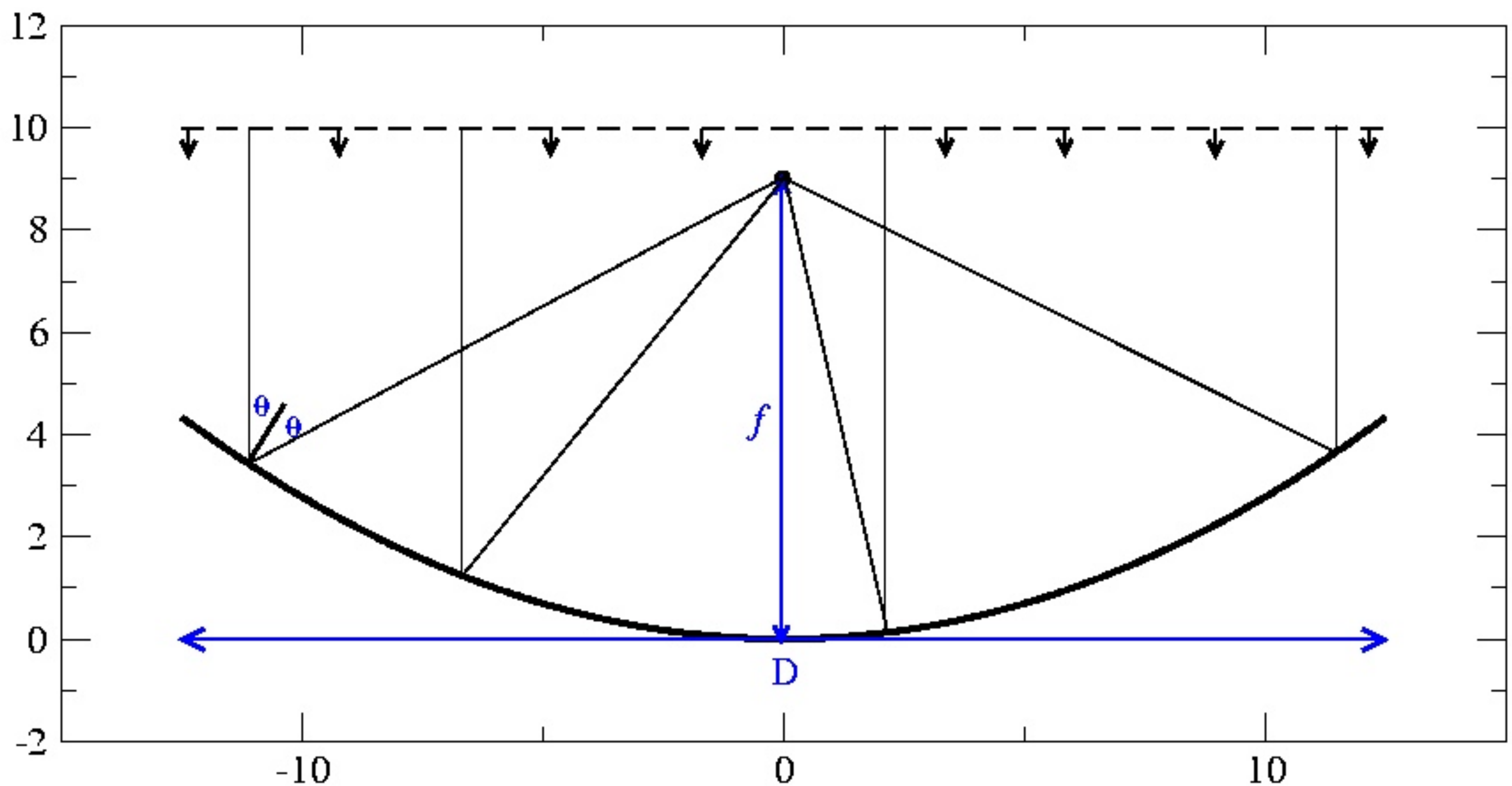
Antennas – the Single Dish

- Antennas span a wide range – from simple elements with nearly isotropic responses, to major mechanical structures designed for high gain and angular resolution.
- The most common antenna is a parabolic reflector – a 'single dish'.
- Understanding how it works will help in our later discussion of interferometry.
- There are four critical characteristics of sensors (antennas):
  1. A directional gain ('main beam')
  2. An angular resolution given by: $\theta \sim \lambda/D$.
  3. The presence of 'sidelobes' – finite response at angles away from the main beam.
- A basic understanding of the origin of these characteristics will aid in understanding the functioning of an interferometer.

The Parabolic Reflector

The parabola has the remarkable property of directing all rays from in incoming wave front to a single point (the focus), all with the same distance. Hence, all rays arrive at the focus with the same phase.

- Key Point: Distance from incoming phase front to focal point is the same for all rays.
- The E-fields will thus all be in phase at the focus – the place for the receiver.



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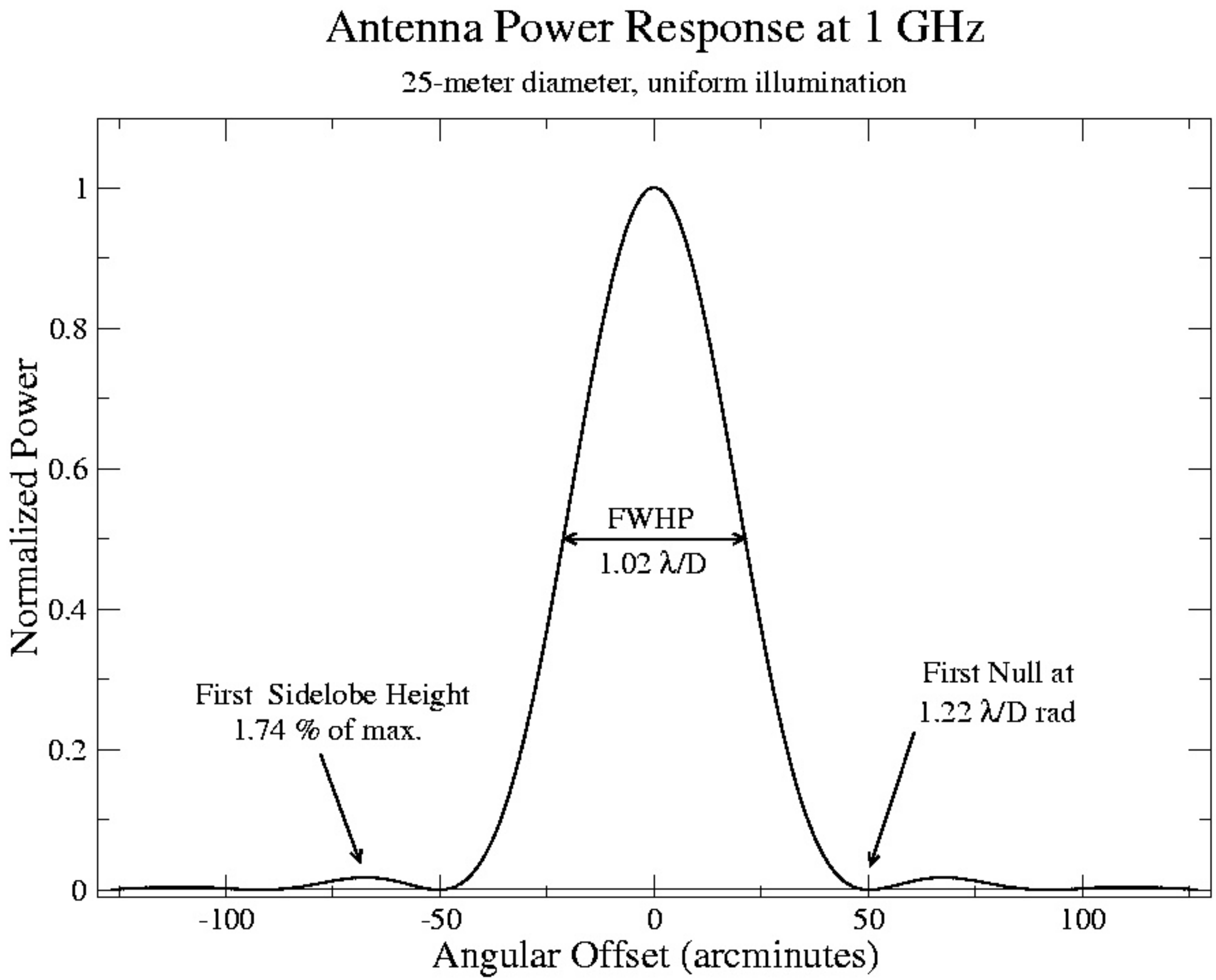
The Standard Parabolic Antenna Response

- The power response of a uniformly
- illuminated circular parabolic antenna of 25-meter diameter,
- at a frequency of 1 GHz.

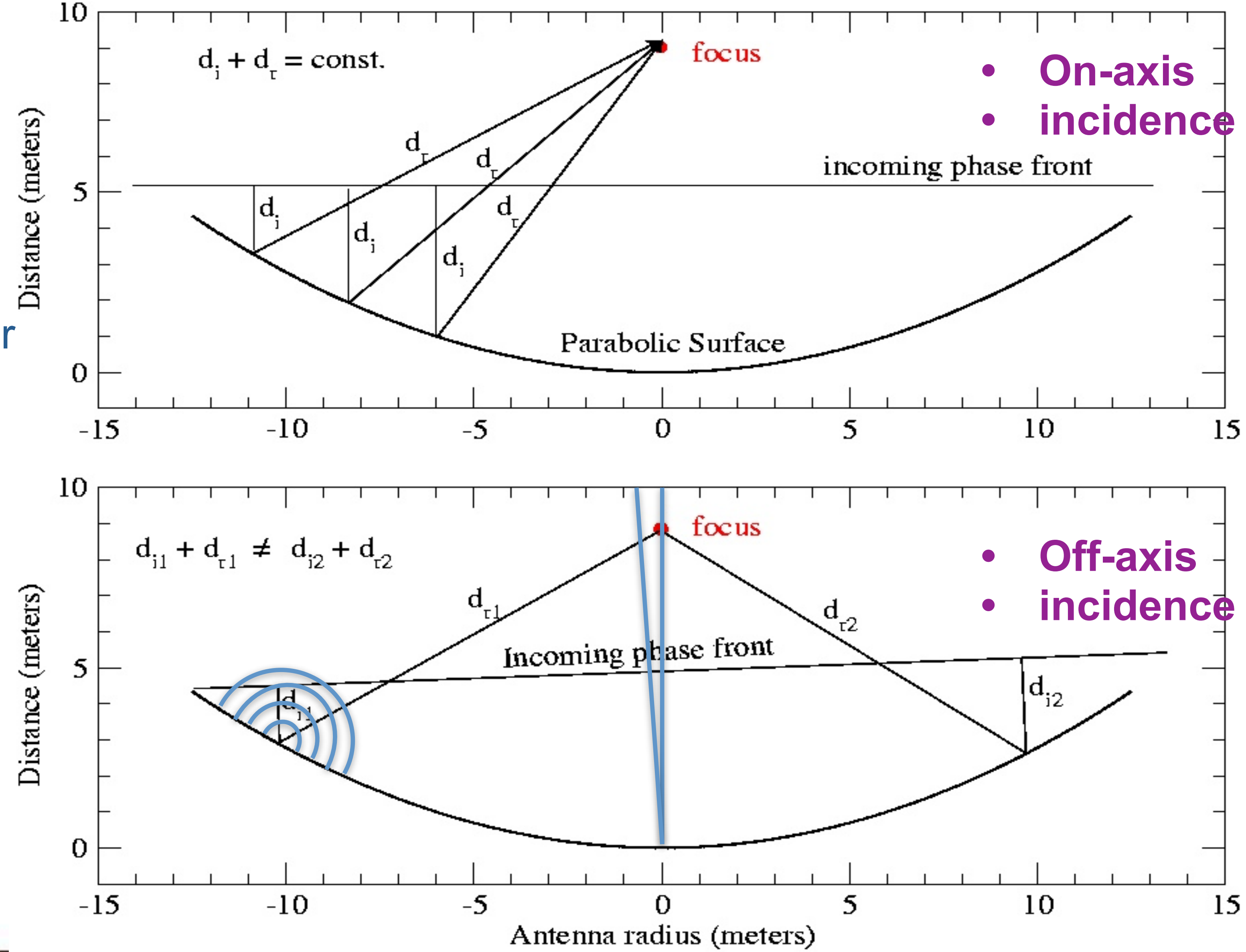


Beam Pattern Origin

- An antenna's response is a result of coherent phase summation of the electric field at the focus.
- First null will occur at the angle where one extra wavelength of path is added across the full width of the aperture:

$\theta \sim \lambda/D$

(Why?)



Specifics: First Null, and First Sidelobe

- When the phase differential across the aperture is 1, 2, 3, … wavelengths, we get a null in the total received power.
  - The nulls appear at (approximately): $\theta = \lambda/D, 2\lambda/D, 3\lambda/D,$ … radians.
- When the phase differential across the aperture is ~1.5, 2.5, 3.5, … wavelengths, we get a maximum in total received power.
  - But, each successive maximum is weaker than the last.
  - These maxima appear at (approximately): $\theta = 3\lambda/2D, 5\lambda/2D, 7\lambda/2D,$ … radians.

Why Interferometry?

- Radio telescopes coherently sum electric fields over an aperture of size D.
- For this, diffraction theory applies – the angular resolution is:

$$\theta_{rad} \approx \lambda / D$$

- In 'practical' units:

$$\theta_{\mathrm{arcsec}} \approx 2\lambda_{\mathrm{cm}} / D_{\mathrm{km}}$$

- To obtain 1 arcsecond resolution at a wavelength of 21 cm, we require an aperture of ~42 km!
- The (currently) largest single, fully-steerable aperture is the 100-m antennas in Bonn, and Green Bank. Nowhere big enough!
- Can we synthesize an aperture of that size with pairs of antennas?
- The methodology of synthesizing a continuous aperture through summations of separated pairs of antennas is called 'aperture synthesis'.

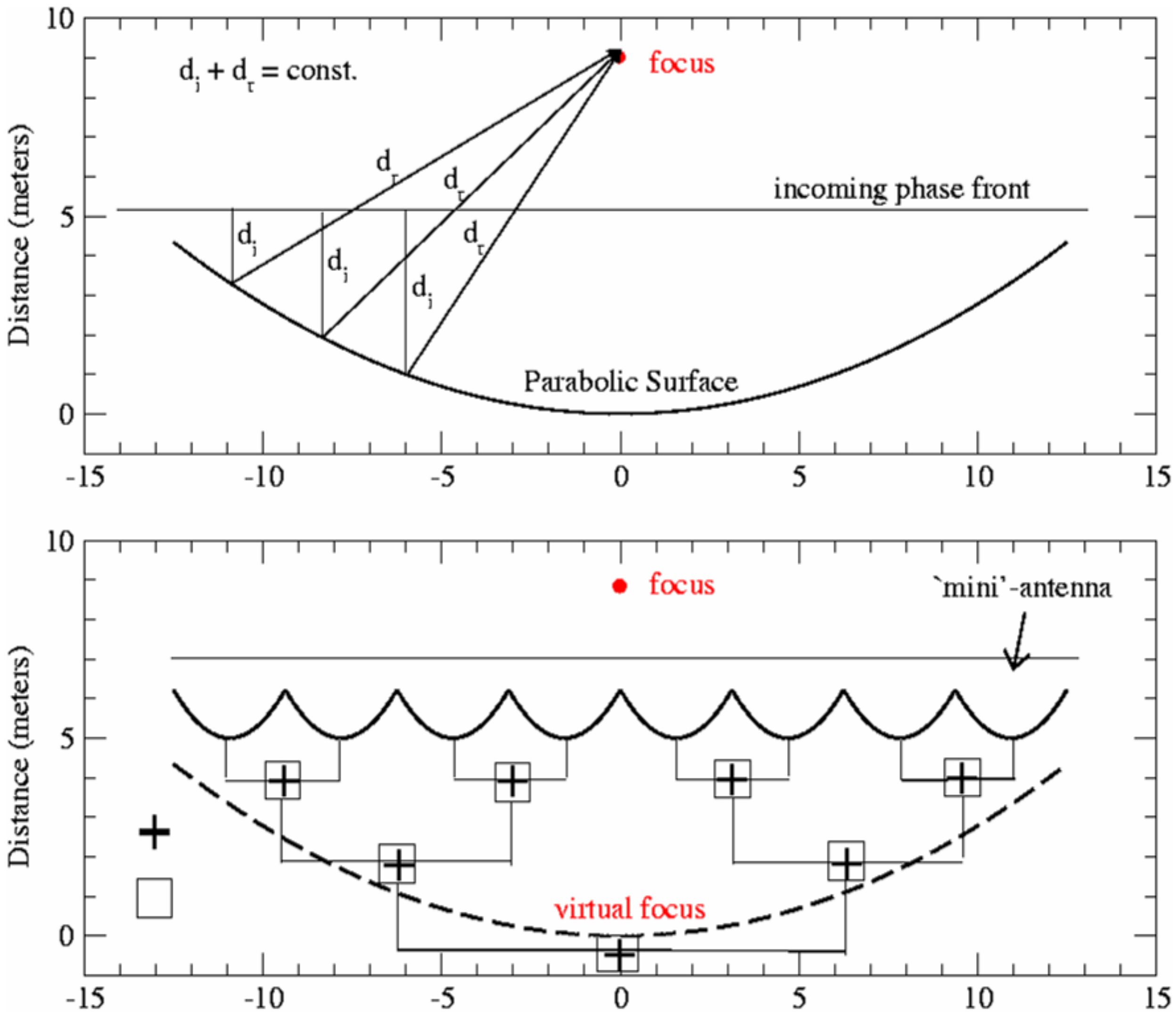
Interferometry – Basic Concept

- We don't need a single parabolic structure!

- We can consider a series of small antennas, whose individual signals are summed in a network.

- This is the basic concept of interferometry.

- Aperture Synthesis is an extension of this concept.



Quasi-Monochromatic Radiation

- Analysis is simplest if the fields are perfectly monochromatic.
- This is not possible – a perfectly monochromatic electric field would both have no power ($\Delta\nu = 0$), and would last forever!
- So we consider instead 'quasi-monochromatic' radiation, where the bandwidth $d\nu$ is finite, but very small compared to the frequency: $\Delta\nu << \nu$
- Consider then the electric fields from a small sold angle $d\Omega$ about some direction **s**, within some small bandwidth $d\nu$, at frequency ν.
- We can write the temporal dependence of this field as:

$$E_\upsilon(t) = E\cos(2\pi\upsilon t + \phi)$$

- The amplitude and phase remains unchanged to a time duration of order $dt \sim 1/d\nu$, after which new values of **E** and ϕ are needed.

Simplifying Assumptions

- We now consider the most basic interferometer, and seek a relation between the characteristics of the product of the voltages from two separated antennas and the distribution of the brightness of the originating source emission.
- To establish the basic relations, the following simplifications are introduced:
  - Fixed in space – no rotation or motion
  - Quasi-monochromatic
  - No frequency conversions (an 'RF interferometer')
  - Single polarization
  - No propagation distortions (no ionosphere, atmosphere …)
  - Idealized electronics (perfectly linear, perfectly uniform in frequency and direction, perfectly identical for both elements, no added noise, …)

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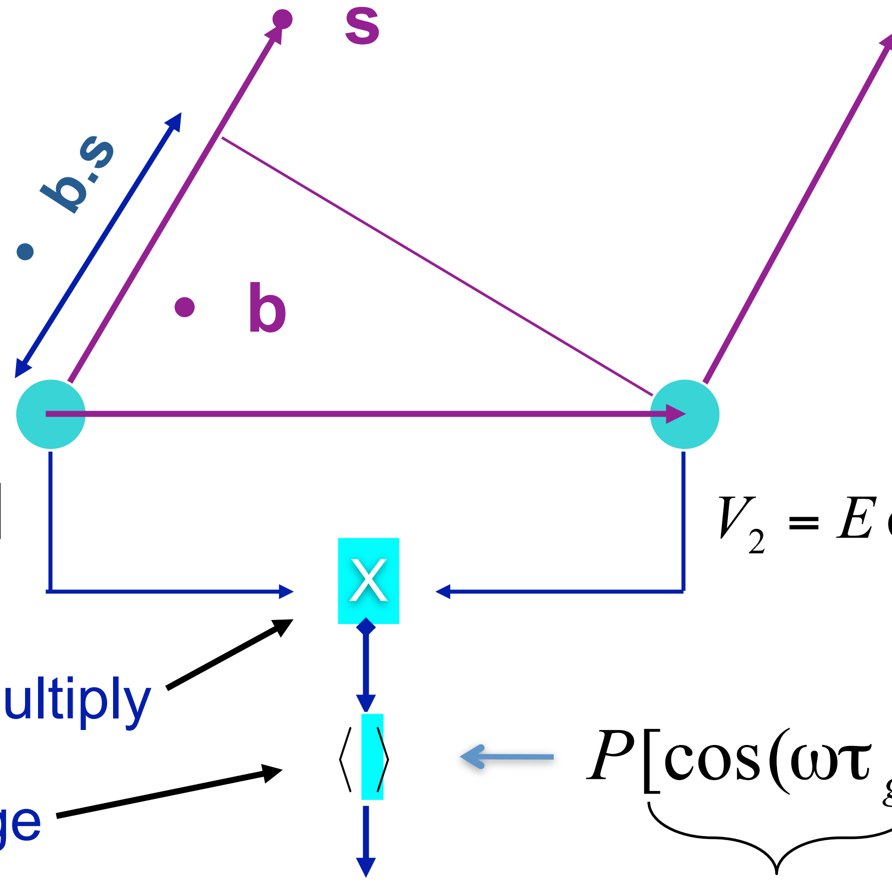
The Stationary, Quasi-Monochromatic Radio-Frequency Interferometer

- Geometric
- Time Delay

$$\tau_g = \mathbf{b}.\mathbf{s}/c$$



$V_1 = E\cos[\omega(t-\tau_g)]$

$V_2 = E\cos(\omega t)$

- The path lengths from sensors
- to multiplier are assumed equal!

- multiply
- average

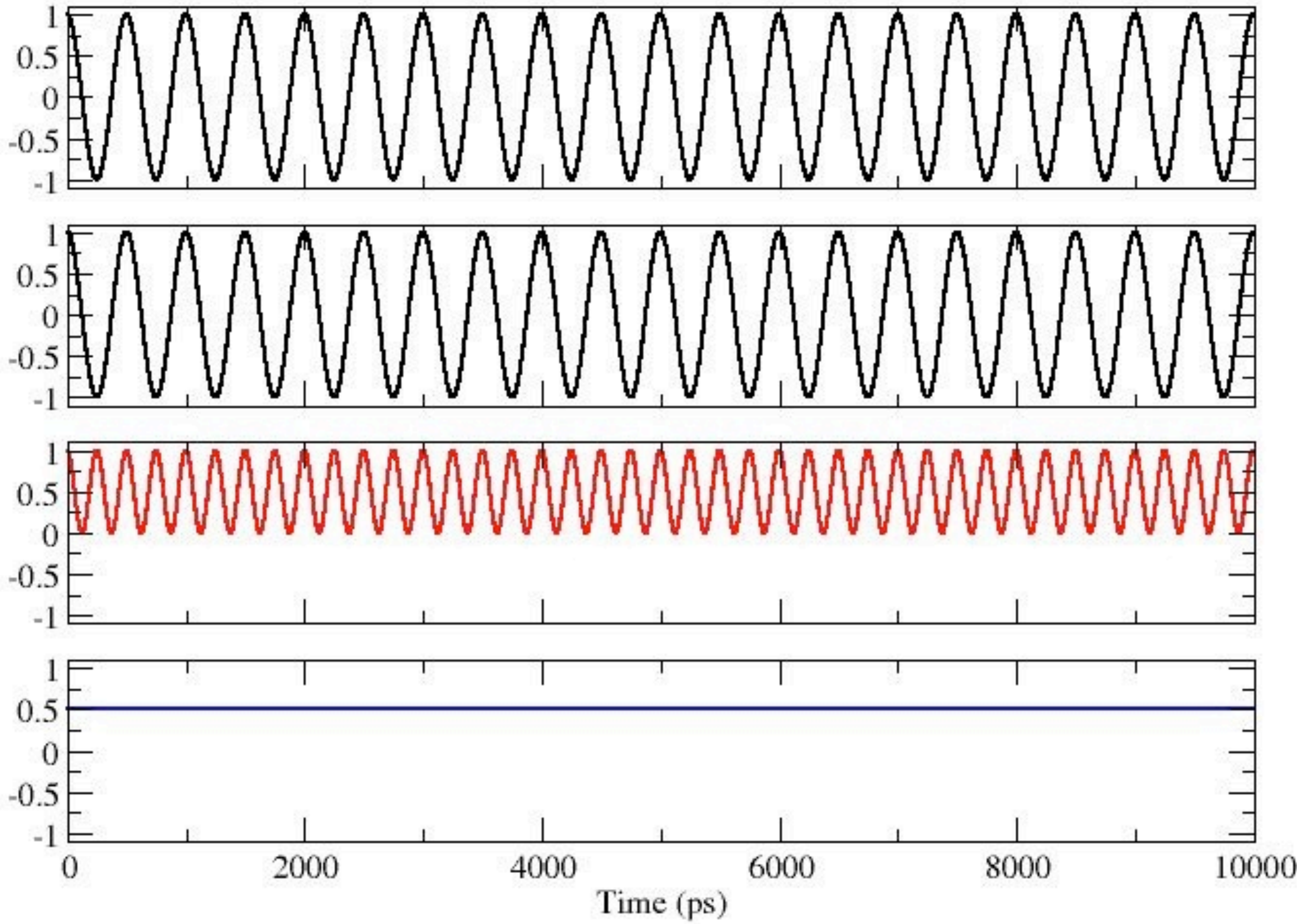
$$P[\underbrace{\cos(\omega\tau_g)}_{\text{Unchanging}} + \underbrace{\cos(2\omega t-\omega\tau_g)}_{\text{Rapidly varying, with zero mean}}]$$

$$R_C = P\cos(\omega\tau_g)$$

Pictorial Example: Signals In Phase

- 2 GHz Frequency, with voltages in phase:
- **b.s** = nλ, or τ_g = n/ν

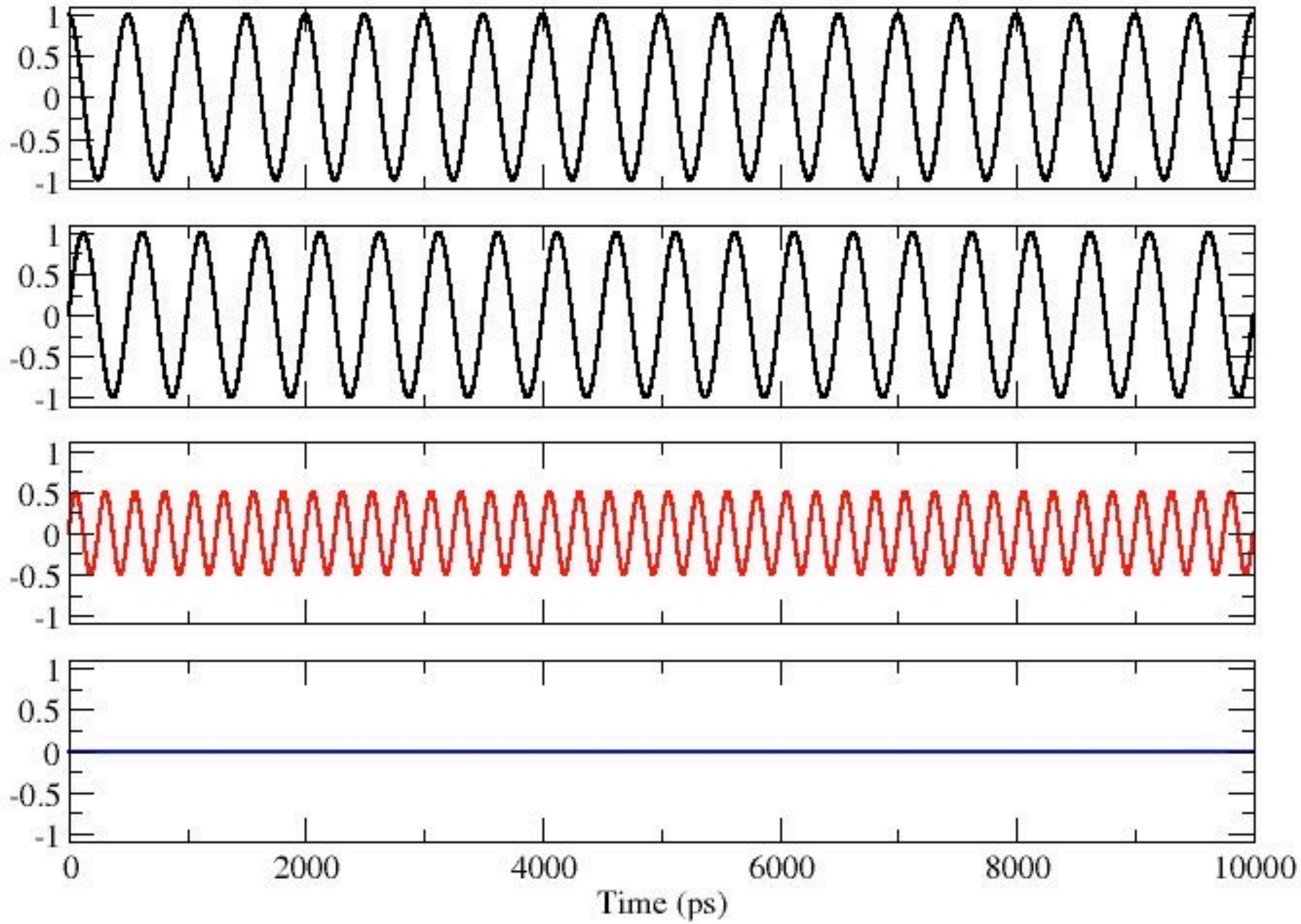
- Antenna 1 Voltage
- Antenna 2 Voltage
- Product Voltage
- Average



Pictorial Example: Signals in Quad Phase

- 2 GHz Frequency, with voltages in quadrature phase:
- $b.s=(n \pm \frac{1}{4})\lambda$, $\tau_g = (4n \pm 1)/4\nu$

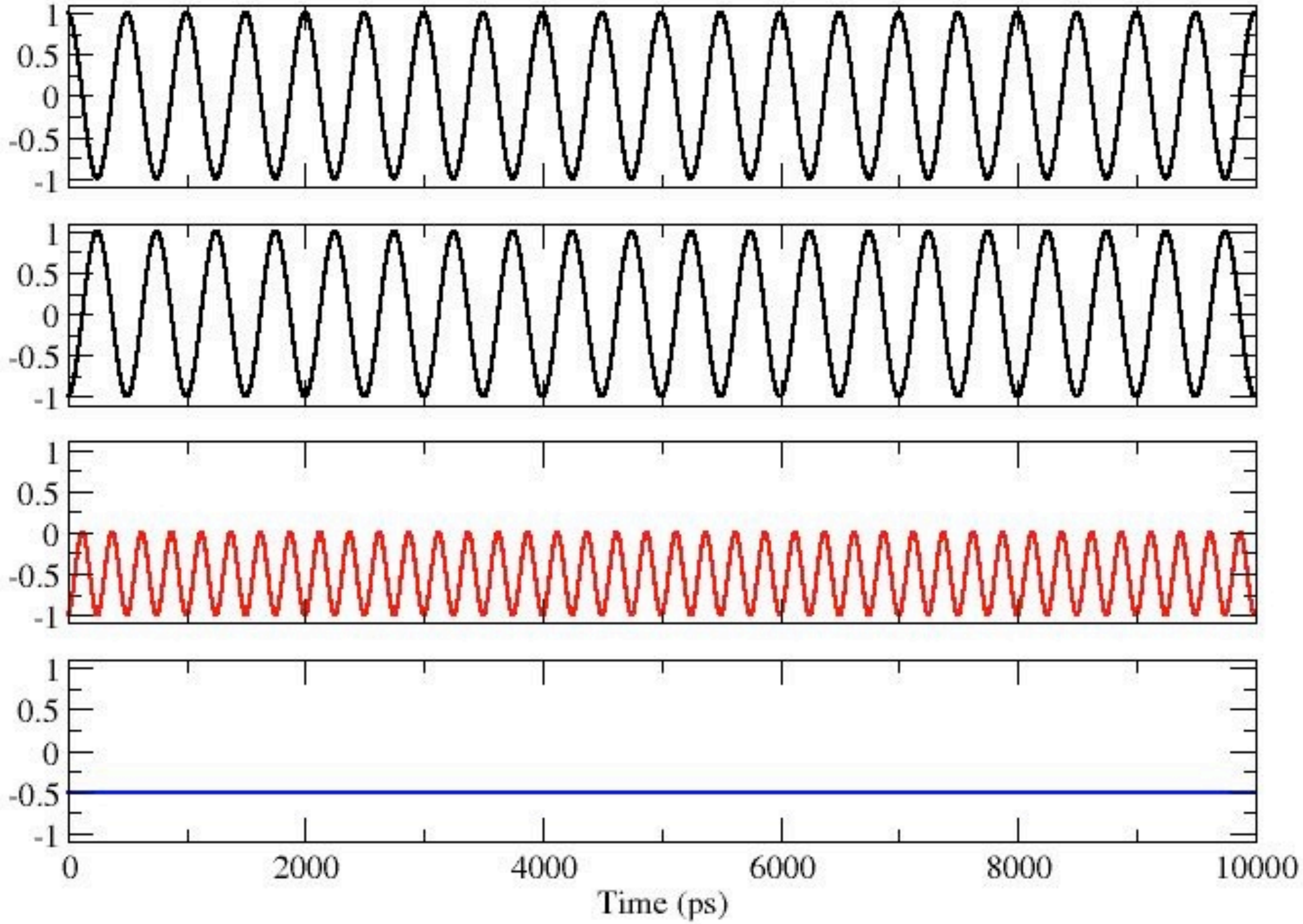
- Antenna 1 Voltage
- Antenna 2 Voltage
- Product Voltage
- Average



Pictorial Example: Signals out of Phase

- 2 GHz Frequency, with voltages out of phase:
- $b.s=(n \pm \frac{1}{2})\lambda \qquad \tau_g = (2n \pm 1)/2\nu$

- Antenna 1 Voltage
- Antenna 2 Voltage
- Product Voltage
- Average



Some General Comments

- The averaged product R_C is dependent on the received power, $P = E^2/2$ and geometric delay, τ_g, and hence on the baseline orientation and source direction:

$$R_C = P\cos(\omega\tau_g) = P\cos\left(2\pi\,\frac{\mathbf{b}\cdot\mathbf{s}}{\lambda}\right)$$

- Note that R_C is not a a function of:
  - The time of the observation -- provided the source itself is not variable!
  - The location of the baseline -- provided the emission is in the far-field.
  - The actual phase of the incoming signal – the distance of the source does not matter, provided it is in the far-field.

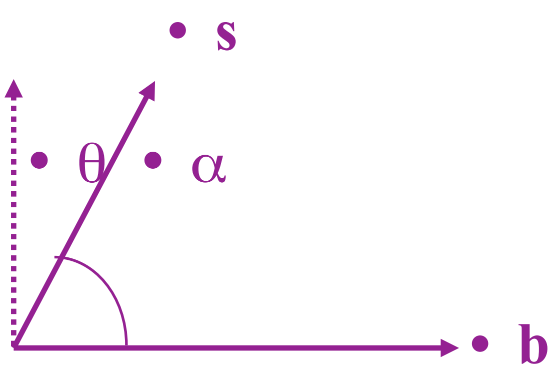
- The strength of the product is dependent on the antenna areas and electronic gains – but these factors can be calibrated for.

Pictorial Illustrations

- To illustrate the response, expand the dot product in one dimension:

$$\frac{\mathbf{b} \bullet \mathbf{s}}{\lambda} = u\cos\alpha = u\sin\theta = ul$$

- Here, **u = b/λ is the baseline length in wavelengths**, and θ is the angle w.r.t. the plane perpendicular to the baseline.
- $l = \cos\alpha = \sin\theta$ is the direction cosine



- Consider the response R_c, as a function of angle, for two different baselines with u = 10, and u = 25 wavelengths:

$$R_C = \cos(2\pi ul)$$

Whole-Sky Response

- Top:

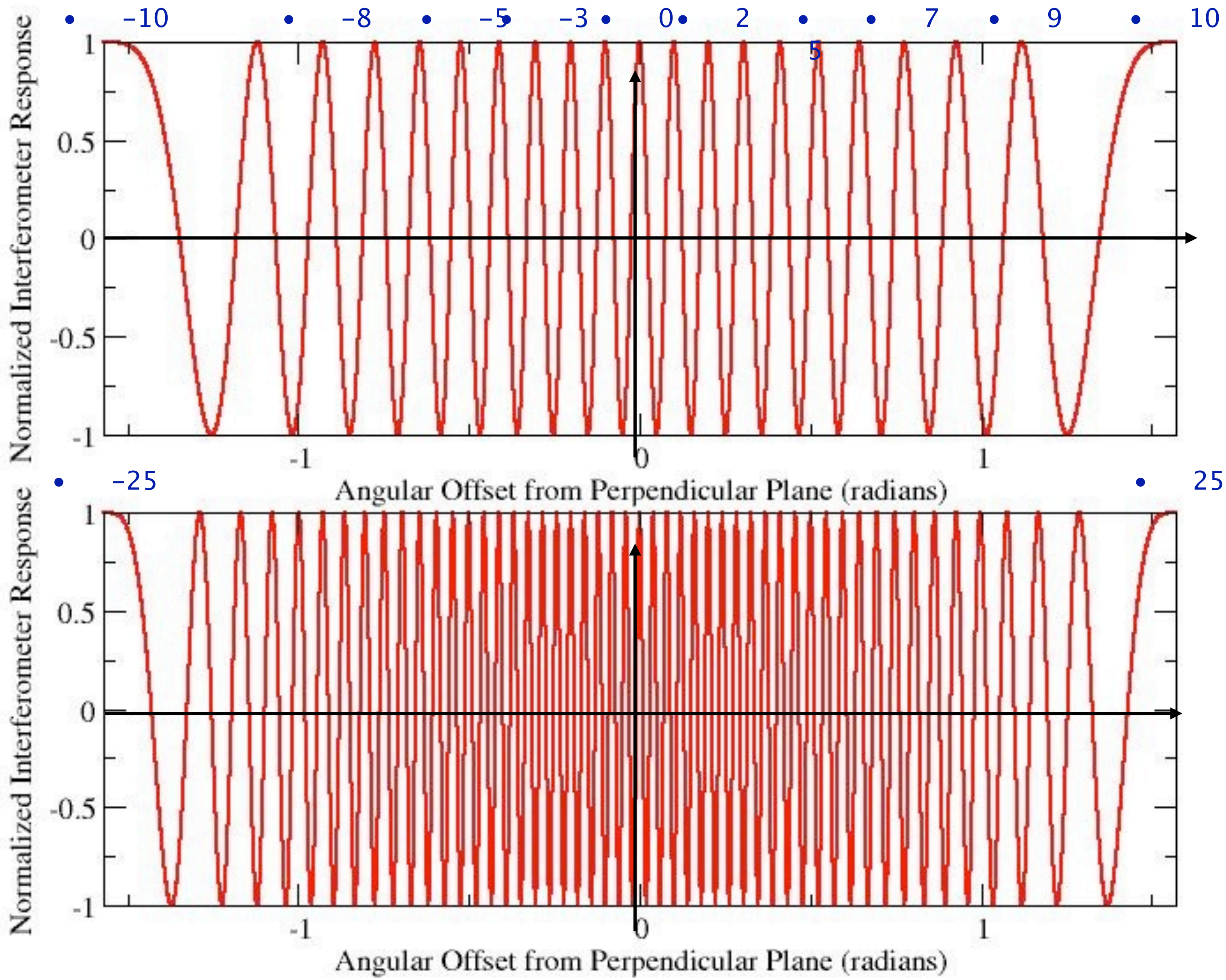
  u = 10

  There are 20 whole fringes over the hemisphere.

- Bottom:

  u = 25

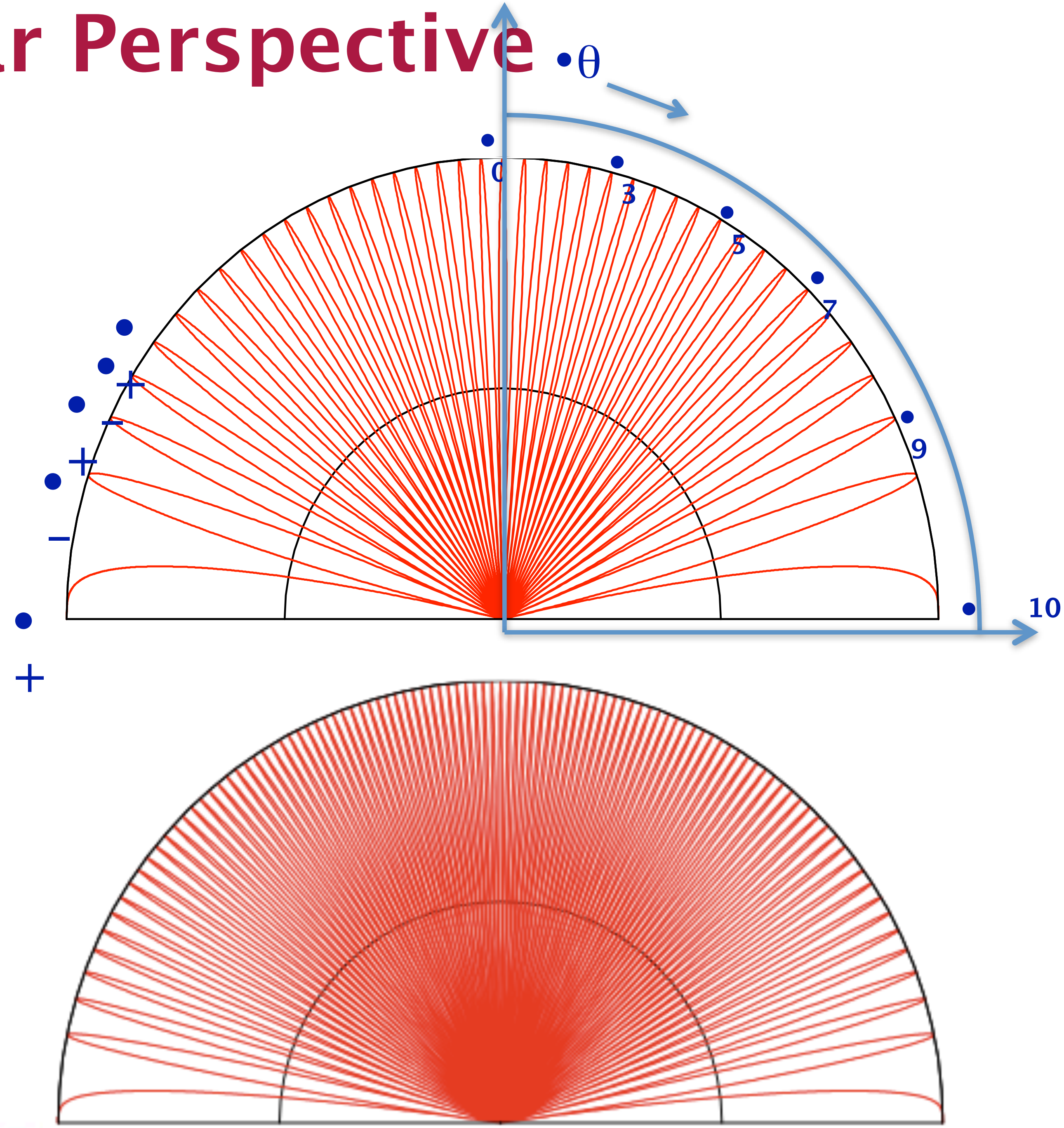
  There are 50 whole fringes over the hemisphere



From an Angular Perspective

- **Top Panel:**
- The absolute value of the response for u = 10, as a function of angle.
- The ‘lobes’ of the response pattern alternate in sign.

- **Bottom Panel:**
- The same, but for u = 25.
- Angular separation between lobes (of the same sign) is
- $\delta\theta \sim 1/u = \lambda/b$ radians.



Hemispheric Pattern

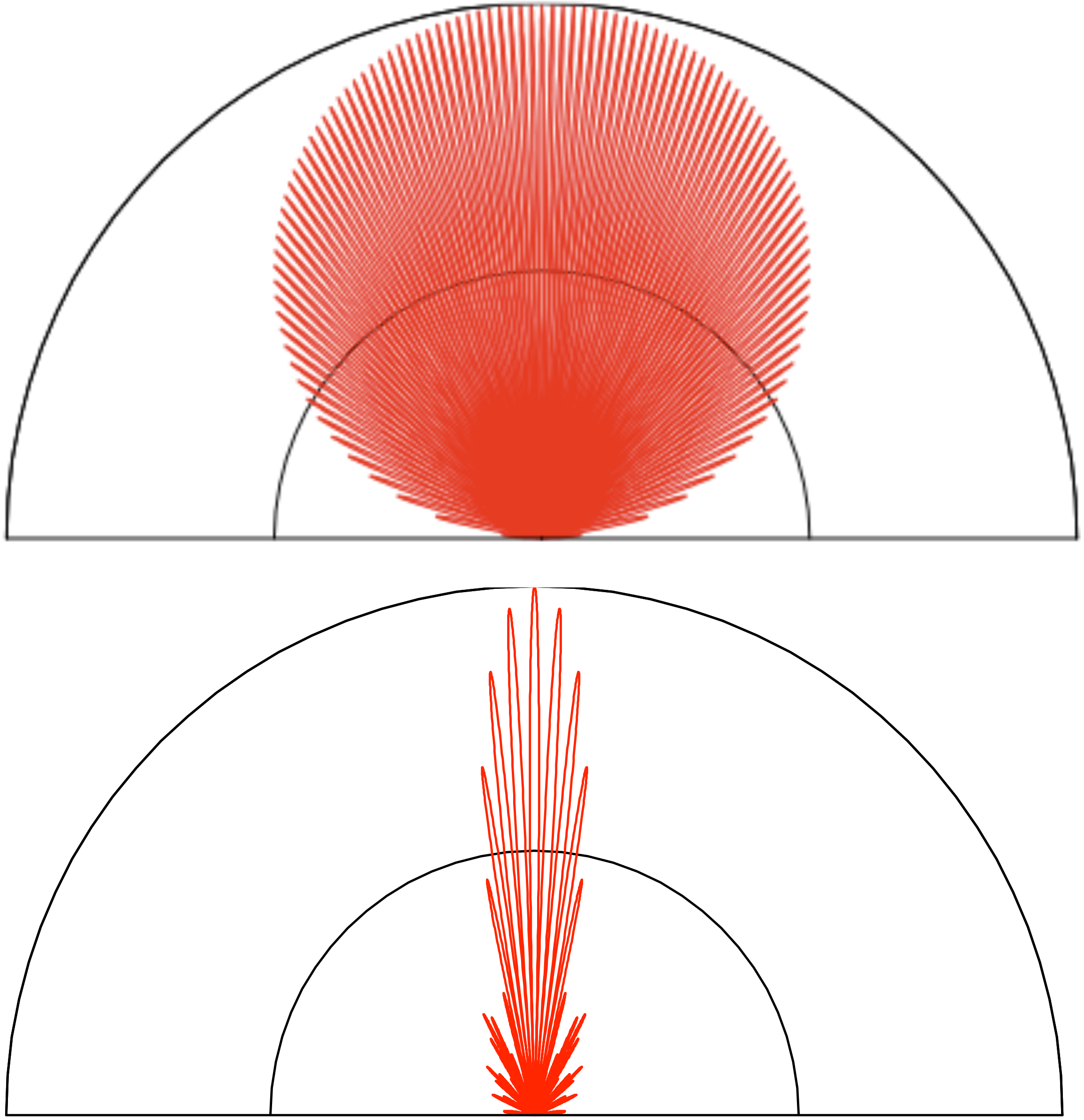
- The preceding plot is a meridional cut through the hemisphere, oriented along the baseline vector.
- In the two-dimensional space, the fringe pattern consists of a series of coaxial cones, oriented along the baseline vector.
- The figure is a two-dimensional representation when u = 4.
- As viewed along the baseline vector, the fringes show a 'bulls-eye' pattern – concentric circles.

The Effect of the Sensor

- The patterns shown presume the sensor has isotropic response.
- This is a convenient assumption, but (sadly, in some cases) doesn't represent reality.
- Real sensors impose their own patterns, which modulate the amplitude and phase, of the output.
- Large sensors (a.k.a. 'antennas') have very high directivity --very useful for some applications.

The Effect of Sensor Patterns

- Sensors (or antennas) are not isotropic, and have their own responses.
- **Top Panel:** The interferometer pattern with a cos(θ)-like sensor response.
- **Bottom Panel:** A multiple-wavelength aperture antenna has a narrow beam, but also sidelobes.



The Response from an Extended Source

- The response from an extended source is obtained by summing the responses at each antenna to all the emission over the sky, multiplying the two, and averaging:

$$R_C = \left\langle \int V_1 d\Omega_1 \int V_2 d\Omega_2 \right\rangle$$

- The averaging and integrals can be interchanged and, **providing the emission is spatially incoherent**, we get

$$R_C = \iint I_\nu(\mathbf{s}) \cos(2\pi\nu\, \mathbf{b} \cdot \mathbf{s}/c)\, d\Omega$$

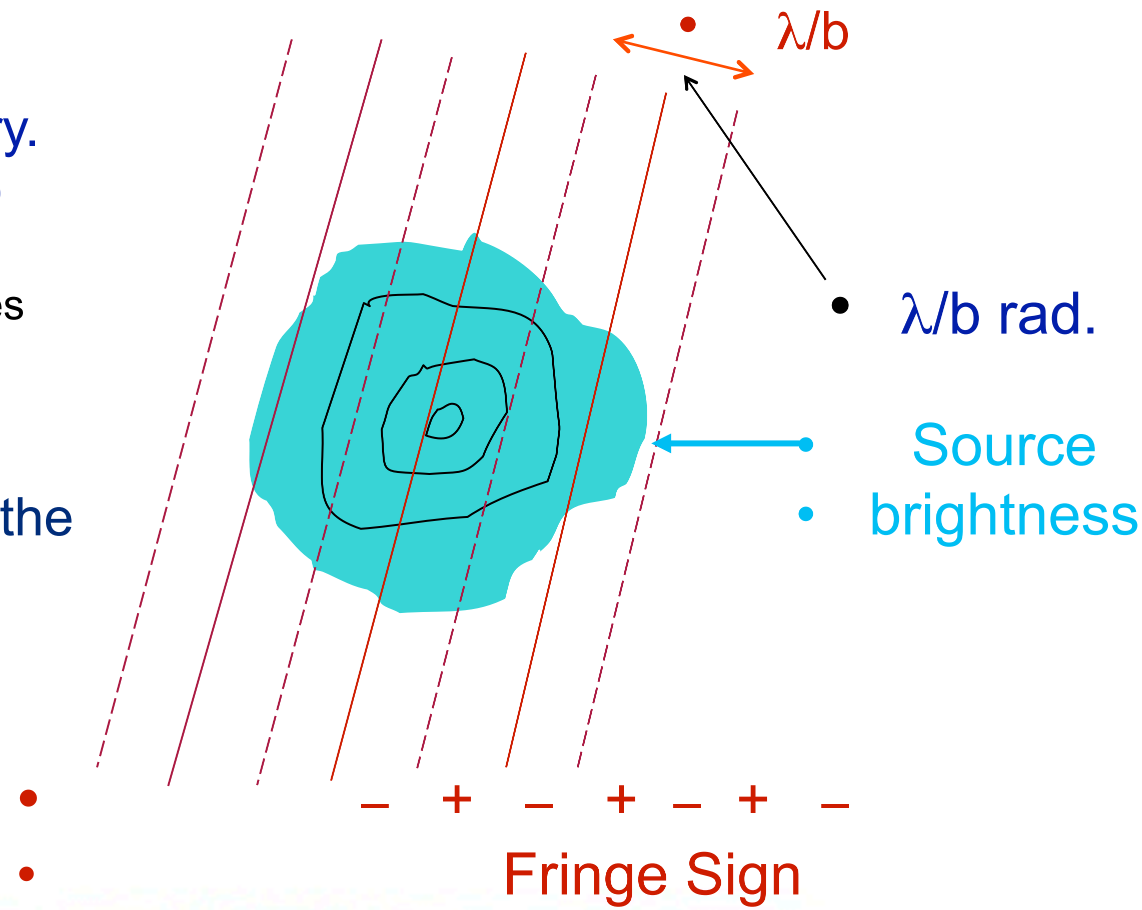
- This expression links what we want – the source brightness on the sky, $I_\nu(\mathbf{s})$, – to something we can measure – R_C, the interferometer response.
- Can we recover $I_\nu(\mathbf{s})$ from observations of R_C?

A Schematic Illustration in 2-D

- The correlator can be thought of ‘casting’ a cosinusoidal coherence pattern, of angular scale ~λ/b radians, onto the sky.

- The correlator multiplies the source brightness by this coherence pattern, and integrates (sums) the result over the sky.

- Orientation set by baseline geometry.
- Fringe separation set by (projected) baseline length and wavelength.
  - Long baseline gives close-packed fringes
  - Short baseline gives widely-separated fringes
- Physical location of baseline unimportant, provided source is in the far field.



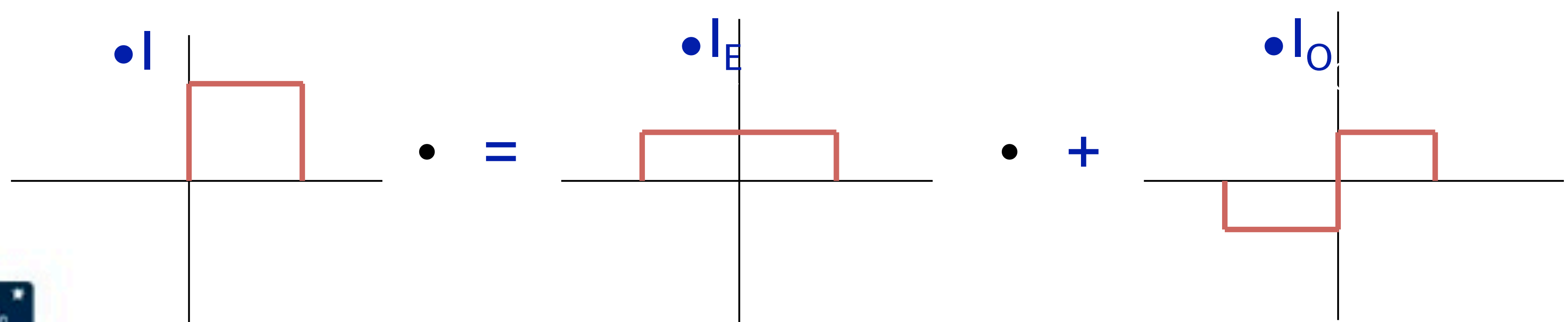
Odd and Even Functions

- Any real function, I(x,y), can be expressed as the sum of two real functions which have specific symmetries:

$$I(x,y) = I_E(x,y) + I_O(x,y)$$

An even part: $$I_E(x,y) = \frac{I(x,y) + I(-x,-y)}{2} = I_E(-x,-y)$$

An odd part: $$I_O(x,y) = \frac{I(x,y) - I(-x,-y)}{2} = -I_O(-x,-y)$$



But One Correlator is Not Enough!

- The correlator response, R_c:

$$R_C = \iint I_\nu(\mathbf{s})\cos(2\pi\nu\,\mathbf{b}\cdot\mathbf{s}/c)\,d\Omega$$

is not enough to recover the correct brightness. Why?

- Suppose that the source of emission has a component with odd symmetry:

$$I_o(s) = -I_o(-s)$$

- Since the cosine fringe pattern is even, the response of our interferometer to the odd brightness distribution is 0!

$$R_C = \iint I_o(\mathbf{s})\cos(2\pi\nu\,\mathbf{b}\cdot\mathbf{s}/c)\,d\Omega = 0$$

- Hence, we need more information if we are to completely recover the source brightness.

Why Two Correlations are Needed

- The integration of the cosine response, R_c, over the source brightness is sensitive to only the even part of the brightness:

$$R_C = \iint I(\mathbf{s})\cos(2\pi\nu\,\mathbf{b}\cdot\mathbf{s}/c)\,d\Omega = \iint I_E(\mathbf{s})\cos(2\pi\nu\,\mathbf{b}\cdot\mathbf{s}/c)\,d\Omega$$

since the integral of an odd function (I_O) with an even function (cos x) is zero.

- To recover the 'odd' part of the intensity, I_O, we need an 'odd' fringe pattern. Let us replace the 'cos' with 'sin' in the integral

$$R_S = \iint I(\mathbf{s})\sin(2\pi\nu\,\mathbf{b}\cdot\mathbf{s}/c)\,d\Omega = \iint I_O(\mathrm{s})\sin(2\pi\nu\,\mathbf{b}\cdot\mathbf{s}/c)\,d\Omega$$

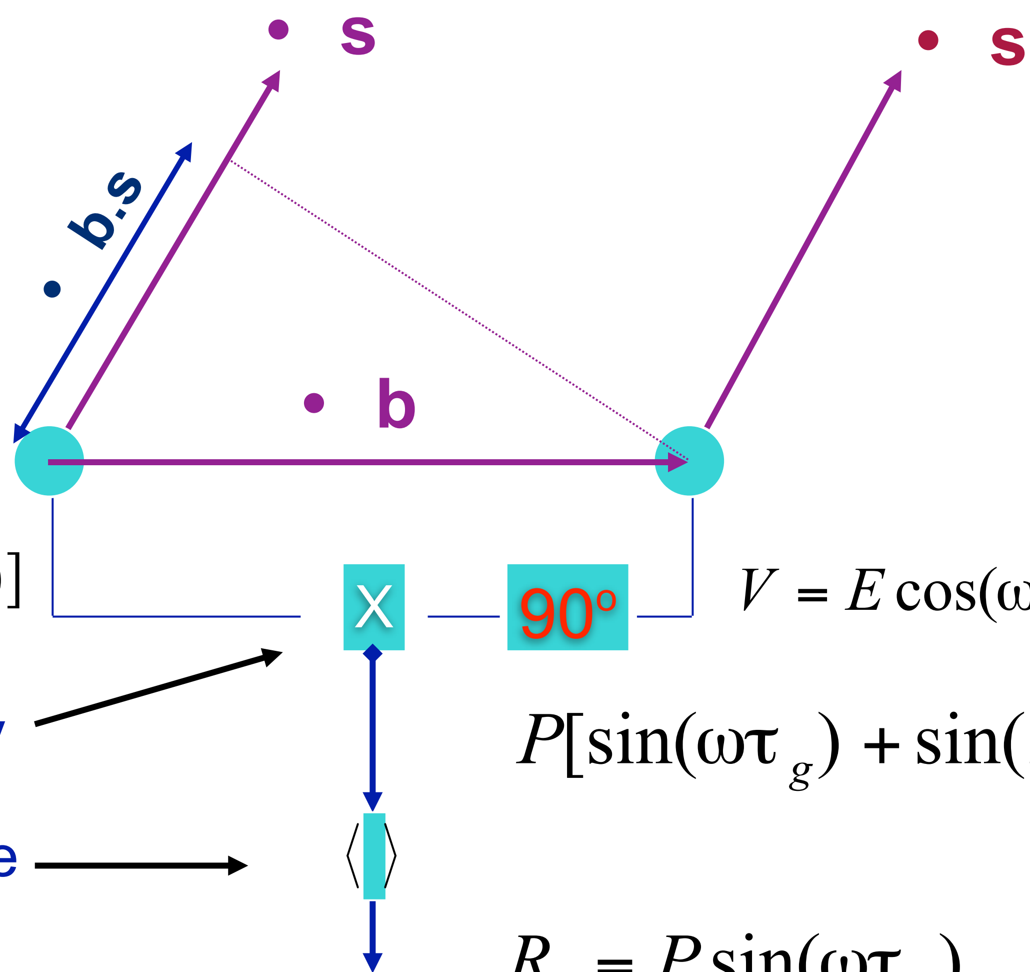
since the integral of an even times an odd function is zero.

- To obtain this necessary component, we must make a 'sine' pattern.

Making a SIN Correlator

- We generate the ‘sine’ pattern by inserting a 90 degree phase shift in one of the signal paths.

$$\tau_g = \mathbf{b}.\mathbf{s}/c$$



$$V = E\cos[\omega(t-\tau_g)]$$

$$V = E\cos(\omega t)$$

- multiply
- average

$$P[\sin(\omega\tau_g) + \sin(2\omega t - \omega\tau_g)]$$

$$R_s = P\sin(\omega\tau_g)$$

Define the Complex Visibility

- We now DEFINE a complex function, the complex visibility, V, from the two independent (real) correlator outputs R_C and R_S:

$$V = R_C - iR_S = Ae^{-i\phi}$$

where

$$A = \sqrt{R_C^2 + R_S^2}$$

$$\phi = \tan^{-1}\left(\frac{R_S}{R_C}\right)$$

- This gives us a beautiful and useful relationship between the source brightness, and the response of an interferometer:

$$V_\upsilon(\mathbf{b}) = R_C - iR_S = \iint I_\nu(s)\, e^{-2\pi i\nu \mathbf{b}\cdot\mathbf{s}/c}\, d\Omega$$

- Under some circumstances, this is a 2-D Fourier transform, giving us a well established way to recover $I(\mathbf{s})$ from $V(\mathbf{b})$.

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The Complex Correlator and Complex Notation

- A correlator which produces both 'Real' and 'Imaginary' parts – or the Cosine and Sine fringes, is called a 'Complex Correlator'
  – For a complex correlator, think of two independent sets of projected sinusoids, 90 degrees apart on the sky.
  – In our scenario, both components are necessary, because we have assumed there is no motion – the 'fringes' are fixed on the source emission, which is itself stationary.
- The complex output of the complex correlator also means we can use complex analysis throughout: Let:

$$V_1 = A\cos(\omega t) = \mathrm{Re}\left(Ae^{-i\omega t}\right)$$

$$V_2 = A\cos[\omega(t - \mathbf{b}\cdot\mathbf{s}/c)] = \mathrm{Re}\left(Ae^{-i\omega(t-b\bullet s/c)}\right)$$

- Then:

$$P_{corr} = \left\langle V_1 V_2^* \right\rangle = A^2 e^{-i\omega\,\mathbf{b}\bullet\mathbf{s}/c}$$

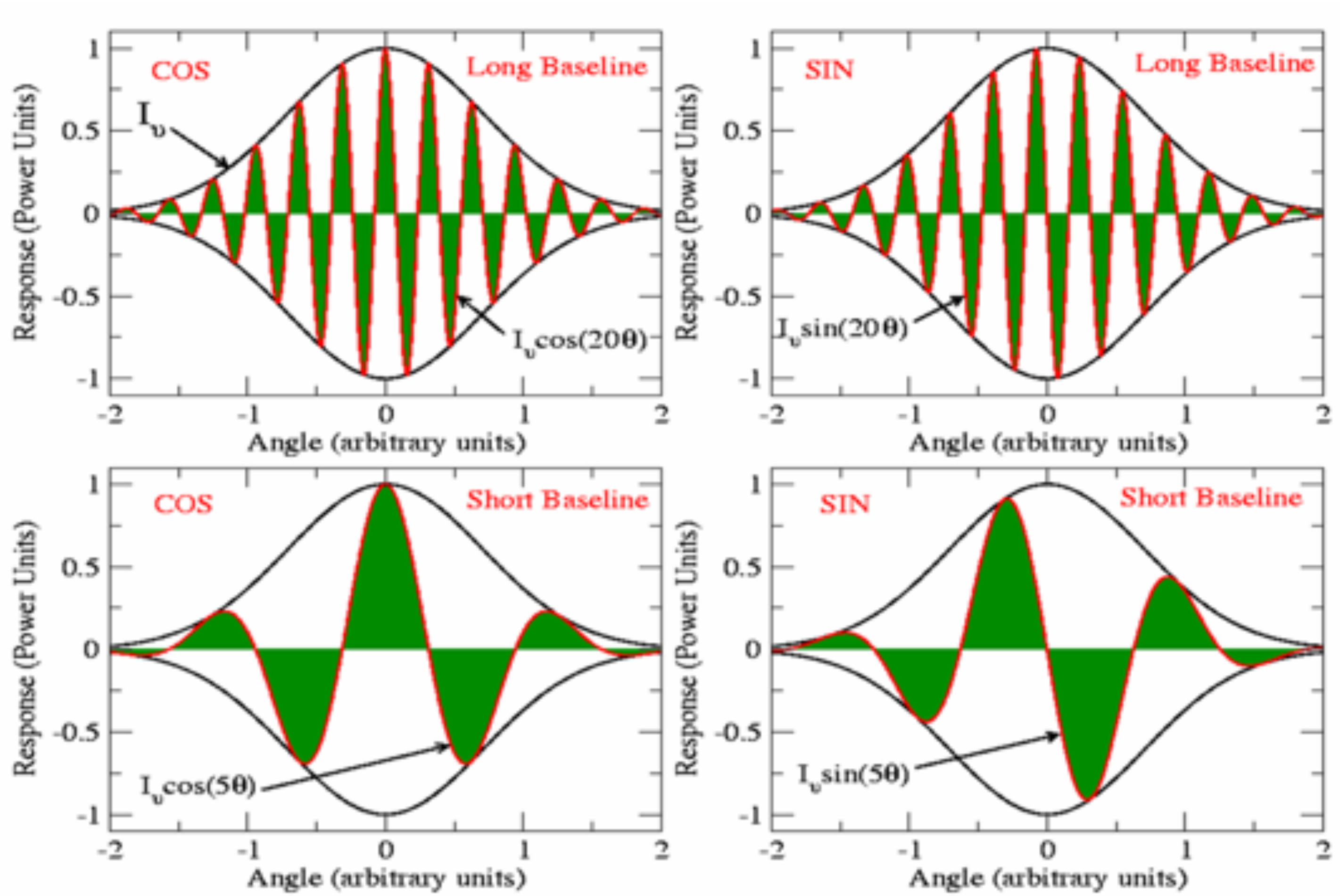
Picturing the Visibility

- The source brightness is Gaussian, shown in black.
- The interferometer ‘fringes’ are in red.
- The visibility is the integral of the product – the net dark green area.

- R_C
- R_S

- Long Baseline

- Short Baseline

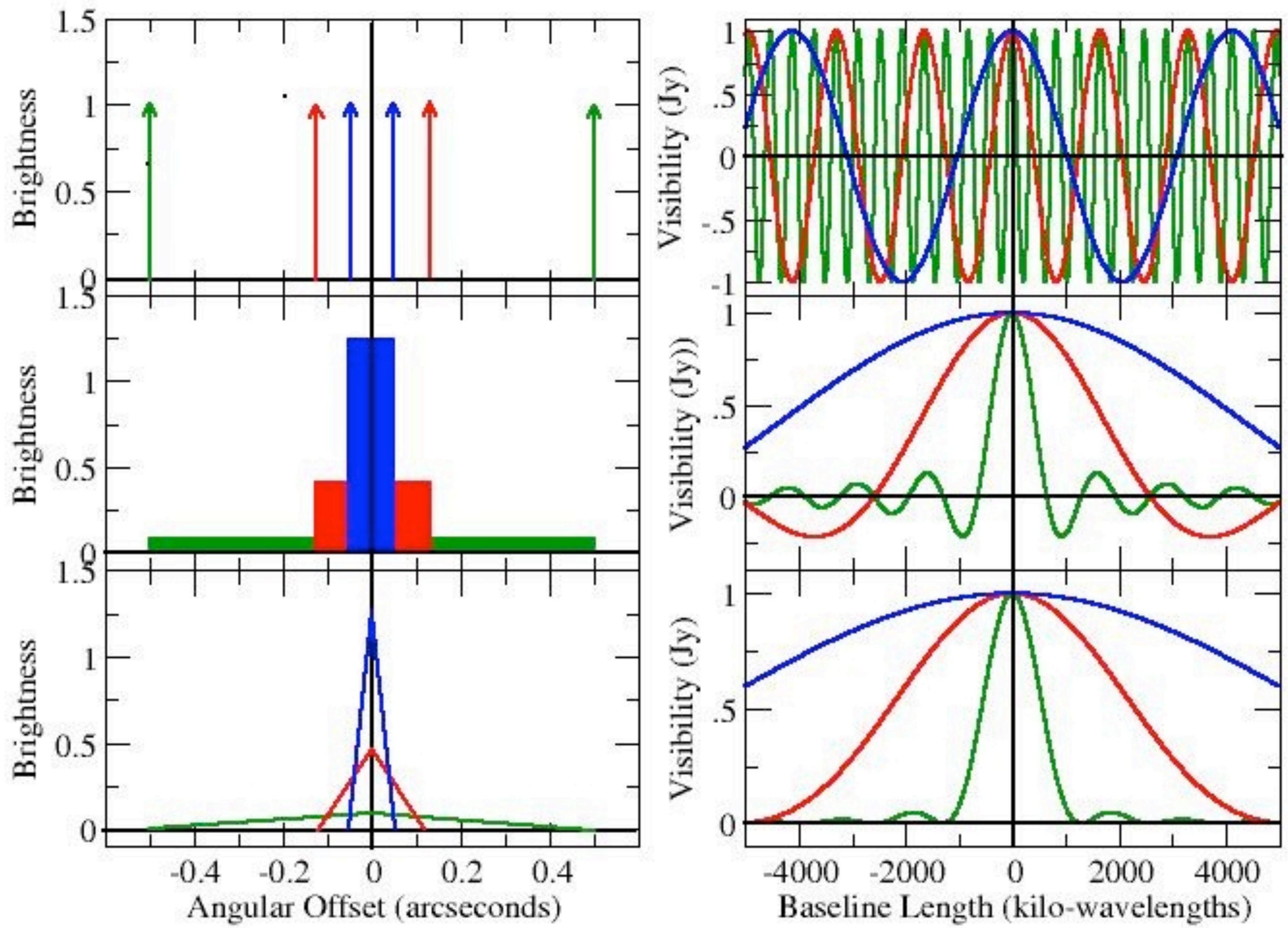


Examples of 1-Dimensional Visibilities

Simple pictures are easy to make illustrating 1-dimensional visibilities.

Brightness Distribution — Visibility Function

- Unresolved Doubles
- Uniform
- Central Peaked



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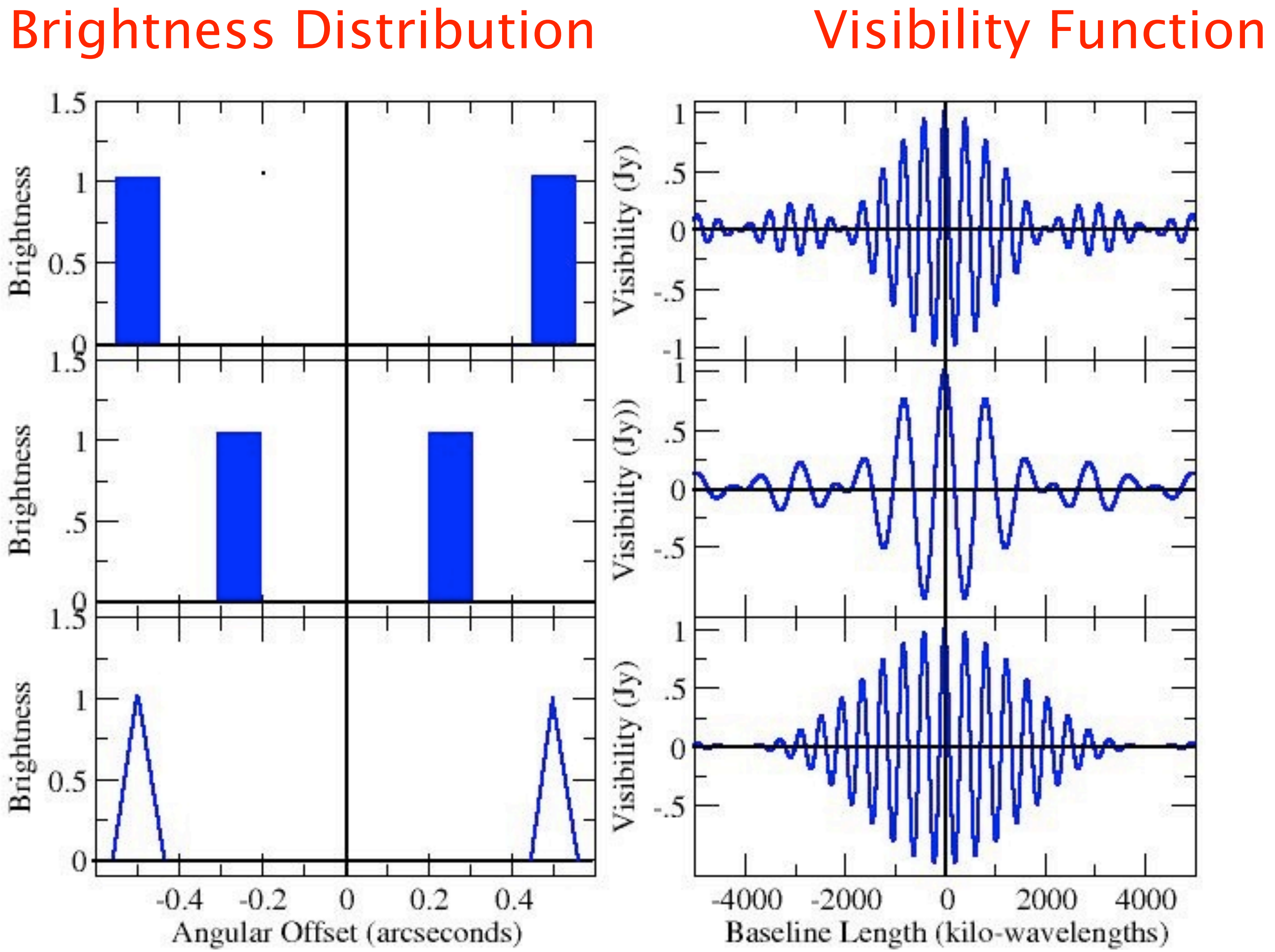
More Examples

Simple pictures are easy to make illustrating 1-dimensional visibilities.

- Resolved Double
- Resolved Double
- Central Peaked Double



Basic Characteristics of the Visibility

- For a zero-spacing interferometer, we get the 'single-dish' (total-power) response.
- As the baseline gets longer, the visibility amplitude will in general decline.
- When the visibility is close to zero, the source is said to be 'resolved out'.
- Interchanging antennas in a baseline causes the phase to be negated – the visibility of the 'reversed baseline' is the complex conjugate of the original.
- Mathematically, the visibility is Hermitian, because the brightness is a real function.

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Some Comments on Visibilities

- The Visibility is a unique function of the source brightness.
- The two functions are related through a Fourier transform. $V_\upsilon(u,v) \Leftrightarrow I(l,m)$
- An interferometer, at any one time, makes one measure of the visibility, at baseline coordinate (u,v).
- Sufficient knowledge of the visibility function (as derived from an interferometer) will provide us a reasonable estimate of the source brightness.
- How many is ‘sufficient’, and how good is ‘reasonable’?
- These simple questions do not have easy answers…

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