

# A solution for galactic disks with Yukawian gravitational potential

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**Abstract** We present a new solution for the rotation curves of galactic disks with gravitational potential of the Yukawa type. We follow the technique employed by Toomre in 1963 in the study of galactic disks in the Newtonian theory. This new solution allows an easy comparison between the Newtonian solution and the Yukawian one. Therefore, constraints on the parameters of theories of gravitation can be imposed, which in the weak field limit reduce to Yukawian potentials. We then apply our formulae to the study of rotation curves for a zero-thickness exponential disk and compare it with the Newtonian case studied by Freeman in 1970. As an application of the mathematical tool developed here, we show that in any theory of gravity with a massive graviton (this means a gravitational potential of the Yukawa type), a strong limit can be imposed on the mass ( $m_g$ ) of this particle. For example, in order to obtain a galactic disk with a scale length of  $b \sim 10$  kpc, we should have a massive graviton of  $m_g \ll 10^{-59}$  g. This result is much more restrictive than those inferred from solar system observations.

## 1 Introduction

General relativity agrees, as is well known, with Newtonian gravitation in the weak field limit (correspondence principle). Some alternatives to general relativity theory, however, not necessary respect this principle. In this way,

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independent of the specific details of the models or theories (for example, scalar-tensor theories of gravity, nonsymmetric gravitational theory, etc.), most of them converge to the same weak field limit: the gravitational potential is Yukawa-like (e.g., Refs. [1–4]).

A first motivation of the present paper is to present a mathematical tool that can be used, independently of the particular theory of gravity, for the study of the rotation curves of galaxies. In this case, there is only one constrain: the potential must be Yukawa-like.

Now, let us start recalling that the Newtonian potential  $\phi$ , as is well known, follows the Poisson equation, namely

$$\nabla^2 \phi = -4\pi G\rho. \quad (1)$$

As a result, the potential of a point mass  $m$  at a distance  $r$  reads

$$\phi = -\frac{Gm}{r}, \quad (2)$$

where  $G$  is the gravitational constant.

Considering a gravitational potential of the Yukawa-type (hereafter named Yukawian gravitation potential), we have for a point mass  $m$  a potential given by

$$\phi = -\frac{Gm}{r} e^{-r/\lambda}, \quad (3)$$

where  $\lambda$  is a constant. Note that, for  $\lambda \rightarrow \infty$  this potential becomes identical to the Newtonian one.

The field equation in this case reads

$$\left(\nabla^2 - \frac{1}{\lambda^2}\right) \phi = -4\pi G\rho. \quad (4)$$

Recall that the constant  $\lambda$  that appears in the Yukawa potential is the Compton wavelength of the exchange particle of mass  $m_g$ . Using this interpretation for the present case one can think of the exchange particle is a massive graviton. In particular, the second motivation of the present paper is to verify if it is possible to constrain the mass ( $m_g$ ) of this particle.

In Sect. 2 we present the new solution for the Yukawian gravitational potential for a thin disk and compare it, in Sect. 3, with the Newtonian solution obtained by [5]. In Sect. 4 we present the main results and conclusions.

## 2 The rotation curve for a thin exponential disk for a Yukawian gravitational potential

Toomre [6] showed that the potential of a thin disk can be written as follows:

$$\phi(r, z) = 2\pi G \int_0^\infty J_0(kr) S(k) e^{-k|z|} dk, \quad (5)$$

where  $S(k)$  is related to the surface density,  $\mu(r)$ , through the following Bessel integral

$$\mu(r) = \int_0^{\infty} J_0(kr)S(k)k dk, \quad (6)$$

where

$$S(k) = \int_0^{\infty} J_0(ku)\mu(u)u du \quad (7)$$

comes from the Fourier–Bessel integral theorem.

It is straightforward to show that Eq. (5) satisfies the Newtonian field equation.

Once  $\mu(r)$  is given one can obtain the potential of the disk, and from the centrifugal-equilibrium condition, namely,

$$g(r) = \frac{v^2(r)}{r} = - \left( \frac{\partial \phi}{\partial r} \right)_{z=0}, \quad (8)$$

one obtains the rotation curve of the disk.

Freeman [5] applied the above equations to obtain the rotation curve for an exponential disk, whose surface density,  $\mu(r)$ , in cylindrical coordinates is given by

$$\mu(r) = \mu_0 e^{-r/b}, \quad (9)$$

where  $b$  is the scale length of the disk.

It is worth stressing that the motivation for adopting an exponential disk comes from observations. Surface photometry shows that the two main components of most spiral and SO galaxies are spheroidal and disk components.

In particular, the radial surface brightness distribution of the disk follows an exponential law, namely,  $I(r) = I_0 e^{-r/b}$ . This implies that the surface density is exponential too, as we are considering.

Finally, the circular velocity, i.e., the rotation curve, for this exponential disk in the Newtonian gravitation reads

$$v^2(r) = \pi G \mu_0 b \left( \frac{r}{b} \right)^2 (I_0 K_0 - I_1 K_1), \quad (10)$$

where  $I$  and  $K$  are the modified Bessel functions, which are calculated at  $r/2b$ .

We now consider how to get a solution for the rotation curve of a thin disk for a Yukawian potential, using the Toomre approach.

The Yukawian potential for a thin disk can be obtained by rewriting Eq. (5) properly, namely,

$$\phi(r, z) = 2\pi G \int_0^{\infty} Z_0 \left( \sqrt{|k^2 - \lambda^{-2}|} r \right) S(k) e^{-k|z|} dk, \quad (11)$$

where

$$Z_0\left(\sqrt{|k^2 - \lambda^{-2}|} r\right) = \begin{cases} J_0\left(\sqrt{|k^2 - \lambda^{-2}|} r\right) & \text{for } k \geq \lambda^{-1} \\ I_0\left(\sqrt{|k^2 - \lambda^{-2}|} r\right) & \text{for } k < \lambda^{-1} \end{cases} \quad (12)$$

As in the Newtonian case the Bessel functions appear in the solution presented above. To account for the extra term “ $\phi/\lambda^2$ ” in the Poisson equation the argument of the Bessel functions must now contain “ $\lambda$ ”.

Obviously, the direct substitution of the above equations satisfy the Yukawian gravitational potential Eq. (4).

We now apply the above equations to obtain the rotation curve for an exponential disk, whose surface density,  $\mu(r)$ , in cylindrical coordinates is given by Eq. (9).

After some manipulation and an eventual change of variables the rotation curve reads

$$v^2(r) = 2\pi G\mu_0 r \left[ \int_{b/\lambda}^{\infty} \frac{\sqrt{x^2 - b^2/\lambda^2}}{(1+x^2)^{3/2}} J_1\left(\frac{r}{b}\sqrt{x^2 - b^2/\lambda^2}\right) dx - \int_0^{b/\lambda} \frac{\sqrt{b^2/\lambda^2 - x^2}}{(1+x^2)^{3/2}} I_1\left(\frac{r}{b}\sqrt{b^2/\lambda^2 - x^2}\right) dx \right] \quad (13)$$

It is worth stressing that we have parameterized the above equation in terms of the ratio “ $b/\lambda$ ”, which helps one to analyze how different a theory with a Yukawian gravitational potential is as compared to the Newtonian theory.

Note that for  $b/\lambda \ll 1$  the second integral of the Eq. (13) goes to zero and the first one becomes

$$\int_0^{\infty} \frac{x J_1\left(\frac{rx}{b}\right)}{(1+x^2)^{3/2}} dx = \frac{1}{2} \frac{r}{b} (I_0 K_0 - I_1 K_1), \quad (14)$$

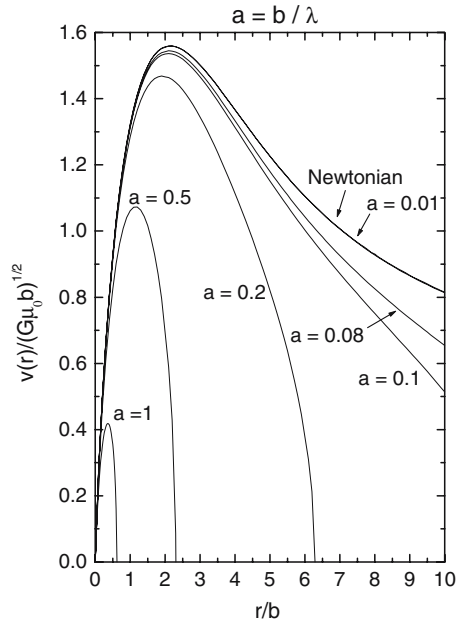
which substituting into Eq. (13) gives, as expected, the Newtonian rotation curve.

In the next section we integrate numerically the Yukawian rotation curve for different values of “ $b/\lambda$ ” and compare them with the Newtonian rotation curve.

### 3 Yukawian versus Newtonian disk

We now compare the Newtonian rotation curve with those that come from a Yukawian gravitational potential. To proceed, however, one needs to integrate Eq. (13), which has three parameters to be specified, namely,  $\lambda$ ,  $\mu_0$  and  $b$ .

**Fig. 1** Comparison between the Newtonian rotation curve with different Yukawian rotation curves parameterized in terms of “ $b/\lambda$ ”



The  $\lambda$  parameter is in principle free, since we are not considering any specific Yukawian gravitational theory. The parameters  $\mu_0$  and  $b$  can be inferred from observations, as already mentioned. For reasons that become clear later on we here only need to know  $b$ .

There is in the literature a series of studies concerning the determination of the disk parameters of spiral and SO galaxies. It is worth noting that there is not any universal value to the scale length  $b$ ; different galaxies present different values for this parameter. From these studies one can conclude that  $b \sim 1 - 10$  kpc [7].

Since  $\lambda$  is a free parameter, whose value depends on the particular theory adopted, we do not fix any particular value for it. Instead, since the scale length is constrained by observations, we present the rotation curves parameterized in terms of “ $b/\lambda$ ”.

In Fig. 1 we compare the Newtonian rotation curve with different Yukawian rotation curves. We plot the velocity, in terms of  $\sqrt{G\mu_0 b}$ , which has dimension of velocity, versus the distance to the center of the disk in units of the scale length  $b$ .

The Newtonian rotation curve has a maximum velocity around  $r/b \sim 2$ , and for larger values of  $r$  the velocity decreases monotonically. This is a well-known result. On the other hand, the behavior of the Yukawian rotation curves is the following: the velocity increases up to a maximum value and then decreases; the greater the ratio “ $b/\lambda$ ”, the lower the maximum velocity is. The size of the disk is strongly dependent on the value of the ratio “ $b/\lambda$ ”. The greater the ratio “ $b/\lambda$ ”, the smaller the disk is. As expected, for  $b/\lambda \ll 1$  the rotation curve becomes Newtonian.

These behaviors of the Yukawian rotation curves strongly constrain the value of  $\lambda$ . As a result, for theories where  $\lambda$  is the Compton wavelength, strong constraints can be imposed on the putative mass of the graviton.

To the Yukawian rotation curve be approximately the Newtonian one, we need  $b/\lambda \ll 1$ . Recall that the Compton wavelength,  $\lambda_c$ , of a particle of mass  $m_g$  reads

$$\lambda_c = \frac{h}{m_g c},$$

where  $c$  is the velocity of light and  $h$  is the Planck constant. For  $b = 10$  kpc we should have

$$m_g \ll 10^{-59} \text{ g.}$$

#### 4 Discussions and conclusions

We present a new solution for the rotation curves of a thin disk for a Yukawian gravitational potential. This thin disk can represent the disks present in spiral and SO galaxies. It is important to stress that the mathematical tool developed here can be used to study rotation curves in any model or theories of gravity which produce a gravitational potential of the Yukawa type.

In particular, we compare the Yukawian rotation curve with the Newtonian one. The Newtonian rotation curve has a maximum velocity and for larger values of  $r$  the velocity decreases monotonically.

The main characteristics of the Yukawian rotation curves are the following: the velocity increases to a maximum value and then decreases; the greater the ratio " $b/\lambda$ ", the lower the maximum velocity is. The greater the ratio " $b/\lambda$ ", the smaller the disk is. As expected for  $b/\lambda \ll 1$  the rotation curve becomes Newtonian.

These behaviors of the Yukawian rotation curves strongly constrain the value of  $\lambda$ . As a result, for theories where  $\lambda$  is the Compton wavelength of a exchange particle, say a graviton, strong constraints can be imposed on its putative mass. For  $b = 10$  kpc, for example, we find  $m_g \ll 10^{-59}$  g.

It is worth stressing that the best bound on the graviton mass from planetary motion surveys is obtained by using Kepler's third law to compare the orbits of Earth and Mars, yielding  $m_g < 10^{-54}$  g [8]. With such a mass and for  $b = 10$  kpc one would obtain that  $b/\lambda \sim 10^5$ , which would imply that galactic discs could not exit (see Fig. 1) for a Yukawian gravitational potential with this value of  $m_g$ .

A study of interest, concerning a Yukawian gravitational potential, would be focused on a complete model for the rotation curve including, besides a disk, a halo. Such a study might impose additional constraints on  $\lambda$  or on the putative mass of a graviton.

Also interesting is to study clusters of galaxies in Yukawian gravitational theory. Since the size of clusters extends to  $\sim 1$  Mpc, strong limits may be imposed

on  $\lambda$  or on the mass of the graviton. We leave these studies, however, to another papers to appear elsewhere.

Although we leave these studies to another papers, our results summarized by the  $a$  parameter in Fig. 1, can already be compared to the galaxy rotation curves of the sample studied by, for example, Ref. [9].

In general, the behavior of the rotation curves, which come from the observations, is the following: the velocity increases as a function of the distance  $r$  to the center of the galaxy; it then becomes nearly flat for large values of  $r$ .

The innermost part of the rotation curves, where the velocity is an increasing function of  $r$ , is dominated by the galactic disk. This very behavior allows us to compare our results with observations. On the other hand, the outermost part of the rotation curve, where it becomes nearly flat, is dominated by the galactic halo (see, e.g., [10, p. 371]).

In particular, Brownstein and Moffat [9] analyzed the rotation curves of a sample of 101 galaxies compiled from [11–16]. All these galaxies present a behavior, in the innermost part of their rotation curves, which are consistent with a Newtonian gravitational potential. There is, as a result, a perfect agreement between our results and the observations if  $a \ll 1$ . This very fact, as already mentioned, strongly constrain the value of the graviton mass.

Moreover, numerous studies of the Tully–Fisher (TF) relations and their applications have been conducted in the past. In most cases their aim was to find observables that reduce the scattering and thus improve this relation as a tool for measuring distances to spiral galaxies, and also to gain insights in the formation and structure of galaxies [17].

In particular, [14] investigates the statistical properties of the TF relations for a volume-limited complete sample of spiral galaxies in the nearby Ursa Major Cluster. They concluded that the TF relation reflects a fundamental correlation between the mass of the halo and the total baryonic mass of the galaxies. This kind of result certainly reinforces our interest to apply the formalism here discussed on a more complete model for rotation curves which includes a massive dark halo.

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