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Class. Quantum Grav. **21** (2004) 4595–4605 PII: S0264-9381(04)80368-X

Polarization states of gravitational waves with a massive graviton

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Received 10 May 2004, in final form 18 August 2004
Published 17 September 2004
Online at stacks.iop.org/CQG/21/4595
doi:10.1088/0264-9381/21/19/008

Abstract
Using the Newman–Penrose formalism, we obtain explicit expressions for the polarization modes of weak, plane gravitational waves with a massive graviton. Our analysis is restricted to a specific bimetric theory whose term of mass for the graviton appears as an effective extra contribution to the stress–energy tensor. For this theory we find that the extra states of polarization have amplitudes several orders of magnitude smaller than purely general relativity (GR) polarizations, h_+ and $h_\times$, in the VIRGO–LIGO frequency band. This result appears using the best limit to the graviton mass inferred from solar system observations and if we consider that all the components of the metric perturbation have the same amplitude h. However, if we consider low frequency gravitational waves (e.g., $f_{\mathrm{GW}} \sim 10^{-7}$ Hz), the extra polarization states produce similar Newman–Penrose amplitudes to purely GR polarization states. This particular characteristic of the bimetric theory studied here could be used, for example, to directly impose limits on the mass of the graviton from future experiments that study the cosmic microwave background (CMB).

PACS numbers: 04.30.−w, 04.80.Nn

1. Introduction

General relativity (GR) assumes that gravitational forces are propagated by a massless graviton. However, the present experimental limits on the mass of the graviton are only based on the behaviour of static gravitational fields as, for example, Newtonian planetary motion in the solar system. If gravity is described by a massive graviton, the Newtonian potential would have Yukawa modifications of the form

$$V(r) = \frac{GM}{r} \exp\left(-r/\lambda_{\mathrm{g}}\right), \qquad (1)$$

where M and λ_g are the mass of the source and the Compton wavelength of the graviton, respectively.

The best bound on the graviton mass from planetary motion surveys is obtained by using Kepler's third law to compare the orbits of Earth and Mars, yielding $m_g < 4.4 \times 10^{-22}$ eV [1]. Another bound on the graviton mass can be established by considering the motions of galaxies in clusters of galaxies, yielding $m_g < 2 \times 10^{-29}$ eV. This second bound is less robust than solar system estimates due to uncertainty about the matter distribution of the universe on large scales [1–3].

A graviton with nonzero mass would produce several effects in the dynamical regime as, for example, extra degrees of polarization for the generation of gravitational waves and velocities of propagation dependent on the frequency of the waves.

Based on these characteristics, [2] has suggested that the mass of the graviton could be bounded using gravitational wave observations. As binary systems evolve, they will slowly spiral together due to the emission of gravitational radiation. Over the course of time, the frequency of the binary orbit rises, ramping up rapidly in the late stages of the evolution, just prior to coalescence. Laser interferometer gravitational wave detectors should be able to track the binary system's evolution, obtaining the detailed time-dependent waveform using the matched filtering techniques required for data analysis in these detectors.

At least in principle, another possibility to identify the effects produced by massive gravitons consists in studying the excited vibrational eigenmodes of spherical gravitational wave detectors to identify the field content of a specific gravitational theory by the observed features of the waves [4].

It is worth stressing that gravitational wave detectors, either interferometers or resonant (cryogenic bars and spheres) are members of a network that will permit the reduction of spurious signals and an experimental determination of the false alarm rate. In addition, three or more detectors ensure the complete reconstruction of a gravitational wave event, including the determination of its velocity of propagation and, the identification of the Riemann tensor signatures.

From the theoretical point of view, massive gravitons have a number of strange properties produced at the level of the field equations. If the mass term has a specific Fierz–Pauli structure [5, 6], the propagator around flat space suffers the van Dam–Veltman–Zakharov (vDVZ) discontinuity [7–9].

On the other hand, the effects of Fierz–Pauli mass terms were considered only around flat space backgrounds. Thus, the discontinuity that was found could be a peculiarity of the flat background and some of the difficulties could be evaded by considering a background with curvature [10].

In fact, it was found that in constant curvature backgrounds (AdS spaces), the extra polarizations of the massive gravitons have a coupling $\sim m_g/H$ where H is the Hubble constant of the AdS space [11, 12]. In this case, the predictions of the massive theory were the same as of the massless theory when $m_g \ll H$.

Nevertheless, it is important to stress that GR has an excellent agreement in the prediction of the decrease of the orbital period (τ) of the binary pulsar PSR 1913 + 16 at least in the weak field limit. The GR predicts $\tau = -2.4 \times 10^{-12}$ and the observed value is $\tau = -(2.40243 \pm 0.00005) \times 10^{-12}$ [3]. Thus, an alternative theory of gravity in the limit $m_g \to 0$ should obtain the same results of GR. Besides, the Newtonian limit has to be valid.

In fact, theories of gravitation can be divided into two different groups: metric and non-metric theories [13]. Basically, a theory of gravitation is said to be metric if it obeys the principle of equivalence. That is, the action of gravitation on the matter is due exclusively to the metric tensor. GR and Brans–Dicke theories are examples of metric theories of gravitation.

Thus, one of the ways to obtain a massive graviton theory, preserving the equivalence principle, is to add a prior geometry. This possibility was recently proposed by [14]. In his theory, it is possible to recover GR when $m_{\rm g} \to 0$ without the problem of the vDVZ discontinuity because the linearized mass term of the theory is not a Pauli–Fierz term.

In the present study, we analyse the polarization states of the gravitational waves, in the theory developed by [14], using the formalism developed by Newman and Penrose [15]. Then, we follow the method proposed by [16, 17] to determine the number of polarization modes of the gravitational waves. The final result consists of explicit expressions for the six independent 'electric' components of the Riemann tensor.

In section 2, we present field equations for the massive graviton using the action proposed by [14]. In section 3, we obtain a general solution for weak gravitational waves with a massive graviton. In section 4, we determine the explicit expressions for the polarization modes of the gravitational waves using the Newman–Penrose formalism. In section 5 we present our conclusions and discuss some aspects of how it could be possible to impose limits on the mass of the graviton from observations of the cosmic microwave background.

2. Field equations for a massive graviton

We wish to examine an extension of linearized GR which includes a massive graviton in the field equations. As explained above, our main idea is to recover GR when $m_{\rm g} \to 0$ so the linearized mass term is not the Pauli–Fierz term. It is an essential condition to have a well-behaved classical limit as the graviton mass goes to zero. According to [14] a way to get this is to add the massive graviton term in the action as an isolated term. That is,

$$I = \frac{c^4}{16\pi G}\frac{I_{\rm G}}{c} + \frac{I_{\rm F}}{c} + \frac{I_{\rm mass}}{c}. \tag{2}$$

The last term on the right-hand side of (2) contributes to the equations of motion with an effective stress tensor given by

$$T^{\mu\nu}_{\rm mass} = \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}} I_{\rm mass}, \tag{3}$$

where

$$T^{\mu\nu}_{\rm mass} = -\frac{m_{\rm g}^2 c^6}{8\pi G\hbar^2}\left\{\left(g_0^{-1}\right)_{\mu\sigma}\left[(g-g_0)^{\sigma\rho} - \frac{1}{2}(g_0)^{\sigma\rho}\left(g_0^{-1}\right)_{\alpha\beta}(g-g_0)^{\alpha\beta}\right]\left(g_0^{-1}\right)_{\rho\nu}\right\}, \tag{4}$$

where $(g_0)_{\alpha\beta}$ is the non-dynamical background metric, $g_{\alpha\beta}$ is the physical (dynamical) metric and $m_{\rm g}$ is the graviton mass.

The weak field limit is obtained with $g_{\alpha\beta} = (g_0)_{\alpha\beta} + h_{\alpha\beta}$ and $|h_{\alpha\beta}| \ll |(g_0)_{\alpha\beta}|$. In reality, as discussed by [14], any algebraic function of the physical metric and background metric with correct linearized behaviour up to second order in h would present the same characteristics of equation (4).

Then, in the weak field limit, $T_{\mu\nu}^{\rm mass}$ becomes

$$T_{\mu\nu}^{\rm mass} = -\frac{m_g^2 c^6}{8\pi G\hbar^2}\left\{h_{\mu\nu} - \frac{1}{2}\left[\left(g_0^{-1}\right)^{\alpha\beta}h_{\alpha\beta}\right](g_0)_{\mu\nu}\right\}. \tag{5}$$

The field equations can be rearranged to produce a structure like the usual Einstein field equations. That is,

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -\frac{8\pi G}{c^4}T^{\mu\nu} - \frac{8\pi G}{c^4}T^{\mu\nu}_{\text{mass}}, \tag{6}$$

where $T^{\mu\nu}$ represents the usual energy–momentum tensor.

From the last equation it is possible to verify that when $m_{\text{g}} \rightarrow 0$, we recover the ordinary Einstein field equations.

3. Weak gravitational waves with a massive graviton

Considering gravitational waves far from field sources, we can work in the weak-field limit. In this case, we should have to choose a particular background geometry for the non-dynamical metric. The natural choice, in a first work, is to take $(g_0)_{\mu\nu}$ to correspond to a flat spacetime (Minkowski spacetime). In principle, this choice should produce good agreement with all astrophysical observations at the level of weak gravitational fields. Thus,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{7}$$

where $|h_{\mu\nu}| \ll 1, \eta_{\mu\nu}$ is the Minkowski spacetime metric and the metric signature is $(-1, +1, +1, +1)$.

Far from gravitational sources, we can take $T^{\mu\nu} = 0$ (vacuum solution) and we have from the conservation of stress–energy

$$\nabla_\mu G^{\mu\nu} = -\frac{8\pi G}{c^4}\nabla_\mu T^{\mu\nu}_{\text{mass}}, \tag{8}$$

where $G^{\mu\nu}$ is the Einstein tensor and ∇ denotes the covariant derivative.

Thus, we have for the right-hand side of equation (8)

$$\frac{m_{\text{g}}^2 c^2}{\hbar^2}\nabla_\nu\left(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h\right) = 0, \tag{9}$$

where we used in the last equation the result obtained from equation (5).

It is worth stressing that equation (9) is not a gauge choice but a constraint imposed by the conservation of energy.

On the other hand, the Ricci tensor is to first order in h

$$R_{\mu\nu} \simeq \frac{\partial}{\partial x^\nu}\Gamma^\alpha_{\alpha\mu} - \frac{\partial}{\partial x^\alpha}\Gamma^\alpha_{\mu\nu} + \text{O}(h^2), \tag{10}$$

and the affine connection is

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\eta^{\alpha\beta}\left[\frac{\partial}{\partial x^\mu}h_{\beta\nu} + \frac{\partial}{\partial x^\nu}h_{\beta\mu} - \frac{\partial}{\partial x^\beta}h_{\mu\nu}\right] + \text{O}(h^2). \tag{11}$$

Thus, equation (6) can be rewritten as

$$\Box\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\right) - m^2\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\right) = 0, \tag{12}$$

where we defined $m^2 = m_{\text{g}}^2 c^2/\hbar^2$.

The last equation, rewritten in terms of the trace reverse of $h_{\mu\nu}$, produces

$$\Box\bar{h}_{\mu\nu} - m^2\bar{h}_{\mu\nu} = 0, \tag{13}$$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - 1/2\eta_{\mu\nu}h$.

The general solution of equation (13) is a linear superposition of solutions of the form

$$\bar{h}_{\mu\nu} = e_{\mu\nu}\exp(\text{i}k_\alpha x^\alpha), \tag{14}$$

where $e_{\mu\nu}$ is the polarization tensor.

From the normalization condition $k_\mu k^\mu = -m^2$, we can obtain the dispersion relation

$$k = \sqrt{(\omega/c)^2 - m^2}. \tag{15}$$

As a consequence of massive gravitons, the speed of propagation of a gravitational wave is dependent on the frequency and given by $v(\omega) = \mathrm{d}\omega/\mathrm{d}k$ which produces

$$v(\omega) = c\sqrt{1 - \frac{m^2c^2}{\omega^2}}. \tag{16}$$

It is important to stress that we are considering in our study a gravitational wave travelling in the $+z$-direction. Thus, the wave vector is given by $k_z = k$ and the metric perturbation presented in equation (14) may be rewritten as

$$\bar{h}_{\mu\nu} = e_{\mu\nu} \exp(-\mathrm{i}\omega t + \mathrm{i}kz). \tag{17}$$

In the next section we will introduce the tetrad formalism to determine the polarization wave modes in this massive theory.

4. Polarization states of gravitational waves with a massive graviton

For nearly null gravitational waves in the weak field limit, it is only necessary to restrict our study to the form and behaviour of the Riemann tensor. It is the Riemann tensor that gives us information about how a gravitational wave interacts with a detector.

To analyse the components of the Riemann tensor into independent wave modes in as invariant a manner as possible, we should investigate the transformation properties of the Riemann tensor under Lorentz transformations which leave the wave direction fixed.

Then, we choose basis vectors in which the components of the Riemann tensor are computed. The basis vectors form a quasiorthonormal tetrad basis. In particular, we follow the formalism derived by [15] and used in the work of [16, 17].

It is worth stressing that the original tetrad formalism differs from the Newman–Penrose formalism only in the manner of choice of the basis vectors. That is, instead of an orthonormal basis, the choice is made of a complex null-basis $(k, l, m, \bar{m})$ where k and l are two real null-vectors and m and $\bar{m}$ are a pair of complex conjugate null-vectors.

These vectors satisfy the orthogonality relations

$$k \cdot l = 1, \qquad m \cdot \bar{m} = -1, \qquad k \cdot m = k \cdot \bar{m} = l \cdot m = l \cdot \bar{m} = 0. \tag{18}$$

In particular, k and l are tangent to the propagation directions of the two plane waves, a wave propagating in the $+z$ direction and a wave propagating in the $-z$ direction, respectively.

Following [16, 17] and taking into account the normalization conditions in equation (18), the vectors $(k, l, m, \bar{m})$ can be written as

$$k = -\tfrac{1}{\sqrt{2}}(1, 0, 0, 1), \tag{19}$$

$$l = -\tfrac{1}{\sqrt{2}}(1, 0, 0, -1), \tag{20}$$

$$m = -\tfrac{1}{\sqrt{2}}(0, 1, \mathrm{i}, 0), \tag{21}$$

$$\bar{m} = -\tfrac{1}{\sqrt{2}}(0, 1, -\mathrm{i}, 0). \tag{22}$$

Although this basis vector forms a null-basis, it is possible to expand the wave vector k for a gravitational wave not exactly null [13], as that obtained from the bimetric theory studied here.

In the Newman–Penrose formalism, the Riemann tensor is split into irreducible parts: the Weyl tensor, the traceless Ricci tensor and the Ricci scalar named, respectively, tetrad components Ψ, Φ, and Λ.

The ten independent components of the Weyl tensor are represented by the five complex scalars,

$$\Psi_0 = -C_{pqrs}k^p m^q k^r m^s, \tag{23}$$

$$\Psi_1 = -C_{pqrs}k^p l^q k^r m^s, \tag{24}$$

$$\Psi_2 = -C_{pqrs}k^p m^q \bar{m}^r l^s, \tag{25}$$

$$\Psi_3 = -C_{pqrs}k^p l^q \bar{m}^r l^s, \tag{26}$$

$$\Psi_4 = -C_{pqrs}l^p \bar{m}^q l^r \bar{m}^s. \tag{27}$$

It is also possible to define the following scalars representing the Ricci tensor

$$\Phi_{00} = -\tfrac{1}{2}R_{kk}, \tag{28}$$

$$\Phi_{01} = \Phi_{10}^* = -\tfrac{1}{2}R_{km}, \tag{29}$$

$$\Phi_{11} = -\tfrac{1}{4}R_{kl} + R_{m\bar{m}}, \tag{30}$$

$$\Phi_{12} = \Phi_{21}^* = -\tfrac{1}{2}R_{lm}, \tag{31}$$

$$\Phi_{22} = -\tfrac{1}{2}R_{ll}, \tag{32}$$

$$\Phi_{02} = \Phi_{20} = -\tfrac{1}{2}R_{mm}, \tag{33}$$

and

$$\Lambda = \frac{R}{24} = \frac{(R_{kl} - R_{m\bar{m}})}{12}. \tag{34}$$

In GR, where $v = c$, only the component Ψ_4 is not null. However, in the bimetric theory studied here, we observe that for gravitational waves with $v(\omega) \lesssim c$, the tetrad components are $\Psi_0 \simeq \mathrm{O}(\varepsilon^2 R)$, $\Psi_1 \simeq \mathrm{O}(\varepsilon R)$, $\Phi_{00} \simeq \mathrm{O}(\varepsilon^2 R)$, $\Phi_{01} \simeq \Phi_{10} \simeq \Phi_{02} \simeq \Phi_{20} \simeq \mathrm{O}(\varepsilon R)$, $\Phi_{11} = 3/2\Psi_2$, and $\Phi_{12} = \Psi_3^*$, where ε is related to the difference in speed between light and the propagating gravitational wave. That is, $\varepsilon = (c/v(\omega))^2 - 1$.

Thus, to describe the six independent components of the Riemann tensor we shall choose the set Ψ_2, Ψ_3, Ψ_4 and Φ_{22}.

It is important to stress that for low frequency gravitational waves where $\varepsilon > 1$, the components Ψ_0, Ψ_1, Φ_{00}, Φ_{01}, Φ_{10}, Φ_{02}, Φ_{20} can be written in functions of Ψ_2, Ψ_3, Ψ_4 and Φ_{22}. Thus, these six independent components are able to describe completely the polarization modes of a gravitational wave.

Then, the explicit expressions for these six components are

$$\Psi_2 = \frac{1}{12}\frac{h_{33}(\omega^4 - \omega^2 k^2) + h_{00}(k^4 - \omega^2 k^2)}{(\omega^2 + k^2)}, \tag{35}$$

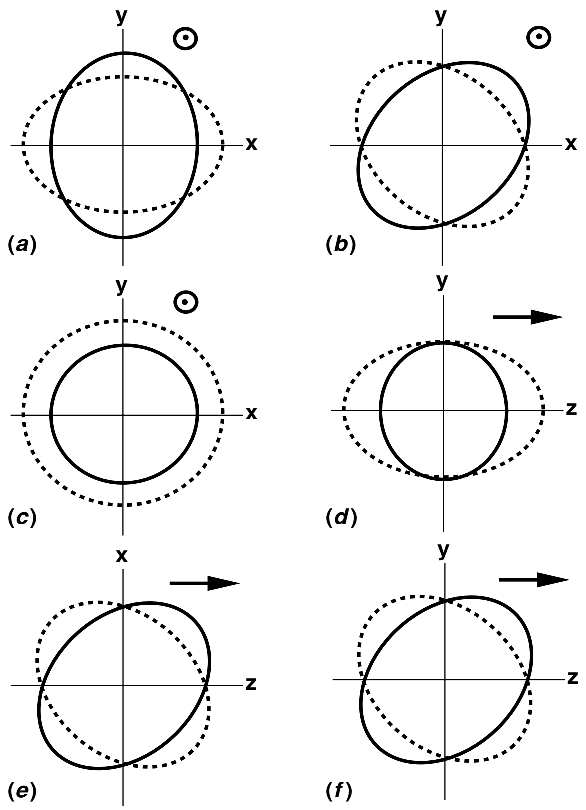

Figure 1. The six polarization modes of weak gravitational waves permitted in any theory of gravity. Shown is the displacement that each mode induces on a sphere of test particles. The wave propagates out of the plane in (*a*), (*b*), (*c*), and it propagates in the plane in (*d*), (*e*) and (*f*). The displacement induced on the sphere of test particles corresponds to the following Newman–Penrose quantities: Re Ψ_4 (*a*), Im Ψ_4 (*b*), Φ_{22} (*c*), Ψ_2 (*d*), Re Ψ_3 (*e*), Im Ψ_3 (*f*).

$$\Psi_3 = \frac{1}{8}\frac{h_{13}(\omega^3 + \omega^2 k - \omega k^2 - k^3) + \mathrm{i}h_{23}(\omega^3 + \omega^2 k - \omega k^2 - k^3)}{\omega}, \tag{36}$$

$$\begin{aligned}\Psi_4 = \frac{1}{8}[&h_{00}(-\omega^4 - 2\omega^3 k + 2\omega k^3 + k^4) - h_{22}(2\omega^4 + 4\omega^3 k + 4\omega^2 k^2 + 4\omega k^3 + 2k^4)\\ &+ h_{33}(-\omega^4 - 2\omega^3 k + 2\omega k^3 + k^4) + \mathrm{i}h_{12}(2\omega^4 + 4\omega^3 k\\ &+ 4\omega^2 k^2 + 4\omega k^3 + 2k^4)]/(\omega^2 + k^2),\end{aligned} \tag{37}$$

$$\Phi_{22} = \frac{1}{8}\frac{(h_{00} + h_{33})(-\omega^4 - 2\omega^3 k + 2\omega k^3 + k^4)}{(\omega^2 + k^2)}, \tag{38}$$

where we use in the above set of equations $c = 1$.

In figure 1 we show the displacement that each mode induces on a sphere of test particles. The wave propagates in the $+z$ direction and has time dependence $\cos\omega t$. The solid line is instantaneous at $\omega t = 0$ and the broken line is one at $\omega t = \pi$.

We can solve the set of equations (35), (36), (37), (38) for determined gravitational wave frequencies in order to identify the contribution of the extra polarization modes for a signal received by a gravitational wave detector.

In particular, we present the results for three different frequencies: $f_{GW} = 100\,\text{Hz}$ that corresponds approximately to the frequency of maximum sensibility for the LIGO interferometer; $f_{GW} = 10^{-3}\,\text{Hz}$ that corresponds to the maximum sensibility for the future laser interferometer space antenna (LISA) and $f_{GW} \simeq 10^{-7}\,\text{Hz}$.

Thus, we have $m_g = 0.44 \times 10^{-21}\text{eV}/c^2$ that corresponds to the best limit from solar system observations [18].

(a) $f_{GW} = 100\,\text{Hz}$:

$$\Psi_2 \simeq 2.1 \times 10^{-35}(h_{33} - h_{00}), \tag{39}$$

$$\Psi_3 \simeq 1.2 \times 10^{-34}(h_{13} + \mathrm{i}h_{23}), \tag{40}$$

$$\Psi_4 \simeq 1.2 \times 10^{-34}(-h_{00} - h_{33}) + 4.4 \times 10^{-16}(-h_{22} + \mathrm{i}h_{12}), \tag{41}$$

$$\Phi_{22} \simeq 1.2 \times 10^{-34}(-h_{00} - h_{33}). \tag{42}$$

(b) $f_{GW} = 10^{-3}\,\text{Hz}$:

$$\Psi_2 \simeq 2.1 \times 10^{-35}(h_{33} - h_{00}), \tag{43}$$

$$\Psi_3 \simeq 1.2 \times 10^{-34}(h_{13} + \mathrm{i}h_{23}), \tag{44}$$

$$\Psi_4 \simeq 1.2 \times 10^{-34}(-h_{00} - h_{33}) + 4.4 \times 10^{-26}(-h_{22} + \mathrm{i}h_{12}), \tag{45}$$

$$\Phi_{22} \simeq 1.2 \times 10^{-34}(-h_{00} - h_{33}). \tag{46}$$

(c) $f_{GW} = 1.1 \times 10^{-7}\,\text{Hz}$:

$$\Psi_2 \simeq -2.1 \times 10^{-36}h_{00} + 3.9 \times 10^{-35}h_{33}, \tag{47}$$

$$\Psi_3 \simeq 7.6 \times 10^{-35}(h_{13} + \mathrm{i}h_{23}), \tag{48}$$

$$\Psi_4 \simeq 8.9 \times 10^{-35}(-h_{00} - h_{33}) + 2.0 \times 10^{-34}(-h_{22} + \mathrm{i}h_{12}), \tag{49}$$

$$\Phi_{22} \simeq 8.9 \times 10^{-35}(-h_{00} - h_{33}). \tag{50}$$

5. Final remarks

Using the bimetric theory proposed by [14], we obtain explicit expressions for the polarization modes of gravitational waves considering a nonzero mass for the graviton. In particular, we analyse what happens to the six tetrad independent components for three different frequencies, namely $f_{GW} = 100\,\text{Hz}$, approximately the frequency of maximum sensibility for the LIGO interferometer, $f_{GW} = 10^{-3}\,\text{Hz}$, the frequency of maximum sensibility for the future laser interferometer space antenna (LISA) and $f_{GW} \simeq 10^{-7}\,\text{Hz}$.

From the above results, we can see that for $f_{GW} = 100\,\text{Hz}$ the dominant tetrad component is Ψ_4, with the part purely GR several orders of magnitude greater than the extra polarization terms. For example, Ψ_4 can be split into $\Psi_4 = (\Psi_4)_{GR} + (\Psi_4)_M$, where the purely GR polarizations are given by $(\Psi_4)_{GR} \simeq 4.4 \times 10^{-16}(-h_{22} + \mathrm{i}h_{12})$ while the term due to the nonzero mass for the graviton is $(\Psi_4)_M \simeq 1.2 \times 10^{-34}(-h_{00} - h_{33})$.

If we consider that all metric perturbations $h_{\mu\nu}$ have a similar value h then we obtain $|(\Psi_4)_{GR}| \simeq 10^{18}|(\Psi_4)_M|$. Thus, the term $(\Psi_4)_M$ has behaviour similar to a very small perturbation when compared to the GR term in Ψ_4. A similar conclusion can be obtained from the comparison of the amplitudes of gravitational waves in equations (39), (40), (41) and (42). We can see that the modes Ψ_2, Ψ_3 and Φ_{22} maintain the same very small perturbation behaviour as the component $(\Psi_4)_{\mathrm{M}}$.

Then, at least for $m_{\mathrm{g}} = 0.44 \times 10^{-21}\mathrm{eV}/c^2$ and taking into account the weak field limit studied in the present paper, it should not be possible to identify the signature of the extra polarization modes from an astrophysical source detected, for example, by the VIRGO and LIGO interferometers. Certainly, as mentioned above, this conclusion is based on the fact that all metric perturbations $h_{\mu\nu}$ have a similar value h. In this case, the extra polarization modes are very small perturbations of $(\Psi_4)_{\mathrm{GR}}$.

For the case $f_{\mathrm{GW}} = 10^{-3}$ Hz, we can see that the extra polarization modes have the same amplitudes as in the frequency $f_{\mathrm{GW}} = 100$ Hz. However, the weight of the term purely GR in Ψ_4 decreases when compared to the massive term $(\Psi_4)_{\mathrm{M}}$. For example, from equation (45) we obtain that $|(\Psi_4)_{\mathrm{GR}}| \simeq 10^8 |(\Psi_4)_{\mathrm{M}}|$ for $f_{\mathrm{GW}} = 10^{-3}$ Hz.

As a consequence, we could think that, in principle, a gravitational wave signal in the LISA frequency range could give us more information on the polarization modes Ψ_2, Ψ_3 and Φ_{22} than a gravitational wave signal in the VIRGO–LIGO frequency range. But unfortunately, as for $f_{\mathrm{GW}} = 10^{-3}$ Hz, the extra polarization modes correspond to very small perturbations of the GR polarizations.

On the other hand, a very interesting result appears for the frequency $f_{\mathrm{GW}} = 1.1 \times 10^{-7}$ Hz. In this case, we can see from equations (47), (48), (49) and (50) that all tetrad components have similar amplitude. As a consequence, we have $|(\Psi_4)_{\mathrm{GR}}| \simeq |(\Psi_4)_{\mathrm{M}}|$. In particular, all the polarization modes produce similar 'excitation' in the frequency 1.1×10^{-7} Hz.

This kind of result is maintained if we change the value of the graviton mass. However, if we reduce the value of the graviton mass, the main result is to shift the frequency f_{c} where all the polarization modes have similar amplitude. That is, the lower the graviton mass, the lower the frequency f_{c} is.

In figure 2 we can see the relation between the mass of the graviton and the frequency f_{c}. For example, if the mass of the graviton is $m_{\mathrm{g}} = 0.44 \times 10^{-21}$ eV$/c^2$, then we have $f_{\mathrm{c}} \simeq 10^{-7}$ Hz. On the other hand, if the mass of the graviton is $m_{\mathrm{g}} = 0.44 \times 10^{-29}$ eV$/c^2$ then $f_{\mathrm{c}} \simeq 10^{-15}$ Hz.

This result could be used to obtain a new limit on the graviton mass based on the analysis of the maps produced by experiments that study the anisotropy of the cosmic microwave background (CMB).

In particular, the generation of a stochastic background of primordial gravitational waves is a fundamental prediction of inflationary models for the early universe. Its amplitude is determined by the energy scale of inflation, which can widely vary between different inflationary models.

Gravitational wave detectors, however, are quite unlikely to have enough sensitivity to detect such a primordial signal, owing both to its smallness and to its extremely low characteristic frequencies. The existence of ultra-low-frequency gravitational radiation, however, can be indirectly probed thanks to the temperature anisotropy and polarization it induces on the CMB radiation.

In particular, the curl component, called B-mode, of the CMB polarization provides a unique opportunity to disentangle the effect of tensor (gravitational-wave) from scalar perturbations, as this is only excited by either tensor or vector modes [19, 20].

From this point of view, future satellite missions, such as Planck, which will have enough sensitivity to either detect or constrain the B-mode CMB polarization predicted by the simplest inflationary models, might represent the first space-based gravitational-wave detector [21].

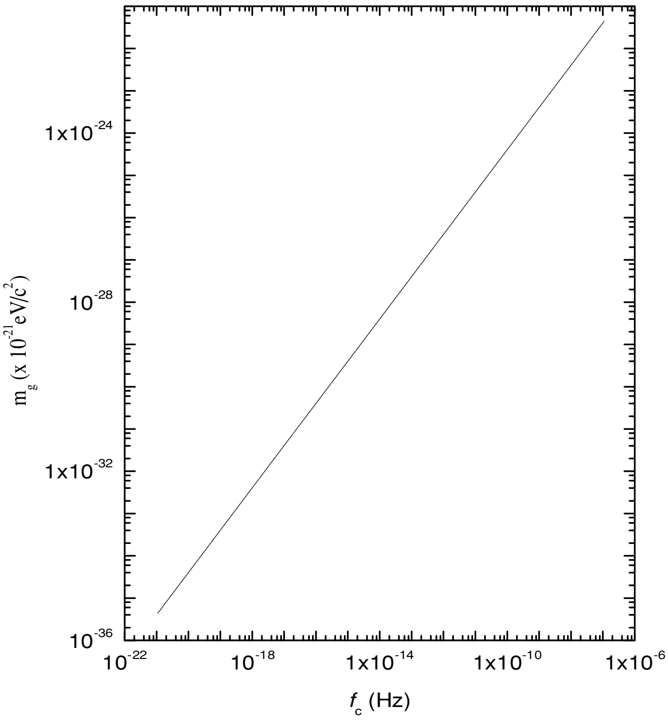

Figure 2. The relation between the mass of the graviton and the frequency where all the polarization modes presented in figure 1 have the same amplitude.

Thus, future CMB missions could present an alternative way to impose a new upper limit on the mass of the graviton and to constrain the number of polarization modes of the gravitational waves.

Acknowledgments

We thank Drs O D Aguiar and J C N de Araujo for stimulating discussions. WLSP and ODM would like to thank the Brazilian agency FAPESP for support (grants 02/13423-2, 02/07310-0 and 02/01528-4, respectively). RMM also thanks the Brazilian agency FAPESP for financial support (grant 99/10809-2).

References

[1] Larson S L and Hiscock W A 2000 *Phys. Rev.* D **61** 4008
[2] Will C M 1998 *Phys. Rev.* D **57** 2061
[3] Will C M 2001 *Living Rev. Rel.* **4** 4
[4] Bianchi M, Coccia E, Colacino C N, Fafone V and Futino F 1996 *Class. Quantum Grav.* **13** 2865
[5] Van Nieuwenhuizen P 1973 *Nucl. Phys.* B **60** 478
[6] Fierz M and Pauli W 1939 *Proc. R. Soc.* A **173** 211
[7] van Dam H and Veltman M J 1970 *Nucl. Phys.* B **22** 397
[8] Zakharov V I 1970 *JETP Lett.* **12** 312
[9] Boulware D G and Deser S 1972 *Phys. Rev.* D **6** 3368
[10] Damour T, Kogan I I and Papazoglou A 2002 *Preprint* hep-th/0212155
[11] Higuchi A 1989 *Nucl. Phys.* B **325** 745

[12] Porrati M 2001 *Phys. Lett.* B **498** 92
[13] Will C M 1993 *Theory and Experiment in Gravitational Physics* (Cambridge: Cambridge University Press)
[14] Visser M 1998 *Gen. Rel. Grav.* **30** 12
[15] Newman E and Penrose R 1962 *J. Math. Phys.* **3** 566
[16] Eardley D M, Lee D L and Lightman A P 1973 *Phys. Rev.* D **8** 3308
[17] Lightman A P and Lee D L 1973 *Phys. Rev.* D **8** 364
[18] Talmadge C, Bertias J P, Hellings R W and Standish E M 1988 *Phys. Rev. Lett.* **61** 1159
[19] Seljak U and Zaldarriaga M 1997 *Phys. Rev. Lett.* **78** 2054
[20] Kamionkowski M, Kosowsky A and Stebbins A 1997 *Phys. Rev. Lett.* **78** 2058
[21] Caldwell R R, Kamionkowski M and Wadley L 1999 *Phys. Rev.* D **59** 027101