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RESEARCH ARTICLE

The conservation of energy–momentum and the mass for the graviton

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Abstract In this work we give special attention to the bimetric theory of gravitation with massive gravitons proposed by Visser in 1998. In his theory, a prior background metric is necessary to take in account the massive term. Although in the great part of the astrophysical studies the Minkowski metric is the best choice to the background metric, it is not possible to consider this metric in cosmology. In order to keep the Minkowski metric as background in this case, we suggest an interpretation of the energy–momentum conservation in Visser's theory, which is in accordance with the equivalence principle and recovers naturally the special relativity in the absence of gravitational sources. Although we do not present a general proof of our hypothesis we show its validity in the simple case of a plane and dust-dominated universe, in which the "massive term" appears like an extra contribution for the energy density.

Keywords Theory of gravitation · Massive graviton · Cosmology

1 Introduction

Could the graviton have a non-zero rest mass? The observations have shown that this is a possibility. One of the most accurate bounding on the mass of the graviton comes from the observations of the planetary motion in the solar system. Variations

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on the third Kepler law comparing the orbits of Earth and Mars can lead us to $m_g < 7.8 \times 10^{-55}$ g [1]. Another bound comes from the analysis of galaxy clusters that lead to $m_g < 2 \times 10^{-62}$ g [2] which is considerably more restrictive but less robust due to the uncertainties in the content of the universe in large scales. Studying rotation curves of galactic disks, [10] has found that we should have a massive graviton of $m_g \ll 10^{-59}$ g in order to obtain a galactic disk with a scale length of $b \backsim 10$ kpc.

The above tests are obtained from static fields based on deviations of the newtonian gravity. In the weak field limit has been proposed [3] to constraint m_g using data on the orbital decay of binary pulsars. From the binary pulsar PSR B1913 + 16 (Hulse–Taylor pulsar) and PSR B1534 + 12 it is found the limit $m_g < 1.4 \times 10^{-52}$ g, which is weaker than the bounds in static field.

It is worth recalling that the mass term introduced via a Pauli–Fierz (PF) term in the linearized approximation produces a theory whose predictions do not reduce to those of general relativity for $m_g \to 0$. This is the so called van Dam Veltmann Zakharov discontinuity [4]. Moreover the Minkowski space as background metric is unstable for the PF theory [5]. However, there is no reason to prefer the PF term over any other non-PF quadratic terms.

It is important to emphasize that these mass terms do not have clear extrapolation to strong fields. A way to do that was proposed by Visser [6]. To generalize the theory to strong fields, Visser makes use of two metrics, the dynamical metric ($g_{\mu\nu}$) and a non-dynamical background metric ($(g_0)_{\mu\nu}$) that are connected by the mass term. Although adding a prior geometry is not in accordance with the usual foundations underlying Einstein gravity, it keeps intact the principles of equivalence (at least in its weak form) and general covariance in the Visser's work. Some interesting physical features emerge from the theory such as extra states of polarizations of the gravitational waves [7].

In the present article, we explore some aspects which are not treated by Visser in his original paper. In the great part of the astrophysical studies the Minkowski metric is the most appropriate choice to the background metric. However, in the study of cosmology, it is not possible to consider this kind of metric, and we need some prior considerations regarding a background metric. Once this problem emerges from the coupling of the two metrics and the energy's conservation condition, we analyze an alternative interpretation of this condition. We also show that this interpretation is in accordance with the equivalence principle and recovers naturally the special relativity in the absence of gravitational sources. Arguments in favor of a Minkowskian background metric in Visser's theory are also considered.

This paper is organized as follows: in Sect. 2 we show how to introduce a mass for the graviton through a non-PF term. We present the strong field extrapolation as given by Visser in Sect. 3. In Sect. 4 we show that the theory is not in accordance with a Minkowski background metric in the study of cosmology. In Sect. 5 we re-interpret the stress–energy conservation in order to keep Minkowski as background in any case. In particular, we show that our re-interpretation is in accordance with the equivalence principle. In Sect. 6 we show why Minkowski is the most natural choice to the background metric. We briefly study some cosmological consequences of our interpretation of the energy–momentum conservation in Sect. 7. And finally, we present our conclusions in the last section.

2 The linearized approximation

The action of a massive gravity in weak field limit may be given by

$$I=\int d^4x\left\{\frac{1}{2}\left[h^{\mu\nu}\Box^2 h_{\mu\nu}-\frac{1}{2}h\Box^2 h\right]-\frac{1}{2}\frac{m_g^2c^2}{\hbar^2}\left[h^{\mu\nu}h_{\mu\nu}-\frac{1}{2}h^2\right]+\frac{8\pi G}{c^4}h^{\mu\nu}T_{\mu\nu}\right\}, \tag{1}$$

where the first term is the linearization of the usual Einstein–Hilbert Lagrangian and the second term is the mass term for the graviton that is a non-PF one. This fact is essential to have a well-behaved classical limit as the graviton mass goes to zero. From Eq. (1) we have the field equation in the weak field regime

$$\Box^2\left[h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h\right]-\frac{m_g^2c^2}{\hbar^2}\left[h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h\right]=-\frac{16\pi G}{c^4}T_{\mu\nu} \tag{2}$$

or

$$\left(\Box^2-\frac{m_g^2c^2}{\hbar^2}\right)\bar{h}_{\mu\nu}=-\frac{16\pi G}{c^4}T_{\mu\nu} \tag{3}$$

where

$$\bar{h}_{\mu\nu}=h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h \tag{4}$$

Equation (3) is a Klein–Gordon type. Note that this equation in the limit $m_g\rightarrow 0$ gives us the weak field equations as in general relativity and the newtonian potential in the non-relativistic limit.

Taking the mass term as above we obtain the condition

$$\partial_\nu\bar{h}^{\mu\nu}=0. \tag{5}$$

as a natural consequence of the energy's conservation [6], instead of a gauge condition as in general relativity. But, as we will see later, this is not the case when one considers strong fields.

3 Visser's strong field equations

Following Visser, the extrapolation of the mass term in Eq. (1) to strong fields could be made by introducing a background metric g_0, which would not be subject to a dynamical equation. So, the mass term of strong fields is given by the action

$$I_{\text{mass}}=\frac{1}{2}\frac{m_g^2c^2}{\hbar^2}\int d^4x\sqrt{-g_0}\left\{(g_0^{-1})^{\mu\nu}(g-g_0)_{\mu\sigma}(g_0^{-1})^{\sigma\rho}(g-g_0)_{\rho\nu}\right.$$
$$\left.-\frac{1}{2}\left[(g_0^{-1})^{\mu\nu}(g-g_0)_{\mu\nu}\right]^2\right\} \tag{6}$$

that recovers the action (1) when we consider the weak field limit:

$$g_{\mu\nu} = (g_0)_{\mu\nu} + h_{\mu\nu}, \quad |h| << 1. \tag{7}$$

Then, the full action considered by Visser is

$$I = \int d^4x \left[\sqrt{-g}\frac{c^4 R(g)}{16\pi G} + \mathcal{L}_{\rm mass}(g, g_0) + \mathcal{L}_{\rm matter}(g) \right] \tag{8}$$

in which the background metric shows up only in the mass term for the graviton. The equations of motion that comes from (8) may be written such as the Einstein equations

$$G_{\mu\nu} = -\frac{8\pi G}{c^4}\left[T_{\mu\nu} + T^{\rm mass}_{\mu\nu}\right], \tag{9}$$

where the contribution of the mass term appears like an extra contribution to the stress–energy tensor, namely

$$\begin{aligned} T^{\mu\nu}_{\rm mass} = -\frac{m_g^2 c^6}{8\pi G\hbar^2} \Bigg\{ \left(g_0^{-1}\right)^{\mu\sigma} \Bigg[& (g - g_0)_{\sigma\rho} \\ & - \frac{1}{2}(g_0)_{\sigma\rho}\left(g_0^{-1}\right)^{\alpha\beta}(g - g_0)_{\alpha\beta} \Bigg] \left(g_0^{-1}\right)^{\rho\nu} \Bigg\}. \end{aligned} \tag{10}$$

Following Eq. (5), the natural extrapolation to strong fields is

$$\nabla_\nu T^{\mu\nu}_{\rm mass} = 0. \tag{11}$$

4 Visser's field equations with a Minkowski background metric

As pointed out by Visser [6], the most sensible choice for almost all astrophysical applications is to choose g_0 as Minkowski. However, some problems appear when we consider this kind of background in cosmology.

To show how these problems emerges, we take the Robertson–Walker as dynamical metric and we consider $k = 0$ for simplicity:

$$ds^2 = c^2dt^2 - a^2(t)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]. \tag{12}$$

To the background metric, we take the following class of metrics:

$$ds_0^2 = b_0^2(t)c^2dt^2 - a_0^2(t)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]. \tag{13}$$

Using these two metrics in the mass tensor and applying (11) we obtain

$$\frac{\dot{a}}{a}\left[\left(\frac{a}{a_0 b_0}\right)^2+\frac{1}{4}\left(\frac{a}{a_0}\right)^4+\frac{1}{4b_0^4}-\frac{1}{b_0^2}\right]+\frac{1}{2}\frac{\dot{a}_0}{a_0}\left(\frac{a}{a_0 b_0}\right)^2$$
$$-\frac{\dot{b}_0}{b_0}\left[\frac{1}{2}\left(\frac{a}{a_0 b_0}\right)^2+\frac{1}{3b_0^4}-\frac{2}{3b_0^2}\right]=0, \tag{14}$$

where dots represent time derivatives.

Thus, $a_0(t)$, $b_0(t)$ and the scale factor $a(t)$ are related to by the differential Equation (14). For example, if we choose the background metric as Minkowski ($a_0=b_0=1$), we obtain that the dynamical metric is Minkowski too. Obviously this is not the case in an expanding Universe, for example.

So, in this case, we cannot consider Minkowski and we need some particular choice to the background metric. Some of these possible choices are discussed in Visser's paper [6].

If a consistent gravitation theory is based on a prior metric, we expect that such a metric would be compatible with any astrophysical case. Once the problems regarding the Minkowskian background metric arises from the condition (11), we will explore an alternative interpretation of the energy–momentum conservation in the remaining of the paper. Such a interpretation has the intention of to keep Minkowski as the background metric in any astrophysical study in Visser's theory.

5 The energy–momentum conservation revisited

From the field equations in Visser's theory we may adopt an alternative energy–momentum conservation condition. Taking the divergence of (9), the left-hand side is a Bianchi identity that is automatically null and from the right-hand side we get

$$\nabla_\nu\left[T^{\mu\nu}+T^{\mu\nu}_{\text{mass}}\right]=0 \tag{15}$$

We will verify if this equation is in accordance with the equations of motion of a free fall test particle describing a geodesic and, therefore, if it is in accordance with the equivalence principle. In the well known Rosen bimetric theory of gravitation [8], for example, it was pointed out the importance of the field equations be in accordance with the geodesic equation which is obtained independently.

To proceed, we adopt the energy momentum tensor to a perfect-fluid:

$$T^{\mu\nu}=(\rho+p)U^\mu U^\nu+pg^{\mu\nu}. \tag{16}$$

Substituting this into Eq. (15) we have

$$\left[(\rho+p)U^\mu U^\nu+pg^{\mu\nu}\right]_{;\nu}=-T^{\mu\nu}_{\text{mass};\nu} \tag{17}$$
$$\left[(\rho+p)U^\nu\right]_{;\nu}U^\mu+(\rho+p)U^\mu{}_{;\nu}U^\nu=-T^{\mu\nu}_{\text{mass};\nu} \tag{18}$$

where ";" denotes the covariant derivative.

Multiplying (18) by U_μ and using

$$U^\mu U_\mu = 1, \tag{19}$$

we obtain

$$\left[(\rho + p)U^\nu\right]_{;\nu} + (\rho + p)U_\mu U^\mu{}_{;\nu} U^\nu = -T^{\mu\nu}_{\text{mass}\ ;\nu} U_\mu. \tag{20}$$

Manipulating (19) we have

$$U^\mu{}_{;\nu} U^\nu = -U^\mu U^\nu{}_{;\nu}; \tag{21}$$

from which we can rewrite Eq. (20) as

$$\left[(\rho + p)U^\nu\right]_{;\nu} - (\rho + p)U^\mu U_\mu U^\nu{}_{;\nu} = -T^{\mu\nu}_{\text{mass}\ ;\nu} U_\mu. \tag{22}$$

From (19) we can find that the second term in the left-hand side of (22) is zero, therefore

$$\left[(\rho + p)U^\nu\right]_{;\nu} = -T^{\mu\nu}_{\text{mass}\ ;\nu} U_\mu. \tag{23}$$

Now substituting (23) in (18) we get

$$-U_\alpha T^{\alpha\nu}_{\text{mass}\ ;\nu} U^\mu + (\rho + p)U^\mu{}_{;\nu} U^\nu = -T^{\mu\nu}_{\text{mass}\ ;\nu}. \tag{24}$$

Since the strong field equations are in accordance with the geodesic equation we have

$$U^\mu{}_{;\nu} U^\nu = 0, \tag{25}$$

which can be rewritten as

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\nu} \frac{dx^\alpha}{d\tau} \frac{dx^\nu}{d\tau} = 0. \tag{26}$$

The last equation can be obtained independently by considering a free fall test particle and the equivalence principle, just as the general relativity theory. Therefore, we conclude that since the energy–momentum conservation condition (15) is in accordance with the equivalence principle, the following relation to the mass term needs to be respected:

$$T^{\mu\nu}_{\text{mass}\ ;\nu} = U^\mu U_\alpha T^{\alpha\nu}_{\text{mass}\ ;\nu}. \tag{27}$$

If we adopt the four-velocity in the rest frame

$$U_\mu = (1, 0, 0, 0), \tag{28}$$

then, we will need to have non-null components of the divergence of the mass term when $\mu = 0$ and $\nu = 0, 1, 2, 3$.

Note that the condition imposed for the mass term (27) is not dependent on the form of the tensor $T^{\mu\nu}_{\text{mass}}$, so the expression (15) is valid to any second rank tensor "interacting" with the perfect fluid.

6 Arguments in favor of a Minkowski background

A classical theory of gravity with a massive graviton apparently needs a background metric for the propagation of this particle. But what is the best physical choice to a background metric? In the Rosen theory [8] the second metric is a flat metric that describes the inertial forces. We will analyze this issue in Visser's theory.

To do that, we take the field Equation (9) in the absence of gravitational source:

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}_{\text{mass}}. \tag{29}$$

In this particular case, following the treatment that we give in this paper, the covariant divergence produces:

$$\nabla_\nu T^{\mu\nu}_{\text{mass}} = 0. \tag{30}$$

Once the mass tensor is constructed by the dynamical metric and by the background metric (and not by derivatives of the metrics), we can conclude that the *most simple way* of satisfying (30) is

$$\nabla_\nu (g_0)^{\mu\nu} = 0 \tag{31}$$

since the divergence of $g_{\mu\nu}$ is null by construction of the covariant derivatives. Then, the natural solution of (31) is

$$(g_0)_{\mu\nu} = g_{\mu\nu}. \tag{32}$$

Which by the construction of the mass term (10) leads to

$$T^{\mu\nu}_{\text{mass}} = 0 \tag{33}$$

and therefore:

$$G_{\mu\nu} = 0. \tag{34}$$

In the absence of gravitational sources the simplest solution of (34) is

$$g_{\mu\nu} = \eta_{\mu\nu} \tag{35}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and by (32) we get

$$(g_0)_{\mu\nu} = \eta_{\mu\nu}. \tag{36}$$

The meaning of our result may be summarized saying that in the absence of gravitational sources the two metrics coincide and we have only one flat metric: Minkowski. In fact this is a simplicity criterion since we expect to recover the results of special relativity in the absence of gravitation. Take, for example, our energy–momentum conservation condition (15), if the background metric is Minkowski, when the dynamical metric is Minkowski too, we get naturally the energy conservation as given in special relativity:

$$\partial_\nu (T^{\mu\nu}) = 0, \tag{37}$$

once the mass term vanishes.

If the background metric is not Minkowski the special relativity is not recovered, because the mass term would not disappear due the coupling of the two metrics.

With all these features the bases of the theory is very close to the foundations of general relativity.

7 Cosmological consequences?

To illustrate the condition (15), let us consider the simple case of matter in the form of an ideal pressure-less fluid, i.e., a cloud of dust particles:

$$T^{\mu\nu} = \rho U^{\mu} U^{\nu}, \tag{38}$$

the Robertson–Walker metric as the dynamic metric and Minkowski as the background one. Then, applying the condition (15) we have

$$\dot{\rho} + \left[3\rho + \frac{3m_g^2 c^6}{16\pi G\hbar^2} \left(4a^2 + a^4 - 3 \right) \right] \frac{\dot{a}}{a} = 0, \tag{39}$$

and Eq. (18) is automatically satisfied.

Solving (39) we obtain the evolution of the energy density as a function of the scale factor:

$$\rho(a) = \frac{\rho_0}{a^3} - \frac{3m_g^2 c^6}{8\pi G\hbar^2} \left(\frac{a^4}{14} + \frac{2a^2}{5} - \frac{1}{2} \right), \tag{40}$$

here, the first term is the evolution of the energy density as calculated in general relativity, and we have an additional term due to the mass term. This may be an interesting treatment of the mass of the graviton in cosmological scenarios, once we can interpret it like a fluid and maybe explain some observational effects that has been attributed to the cosmological constant, quintessence and other exotic fluids [9].

Another interesting feature emerges from our treatment. It is not possible to obtain a de Sitter solution for the vacuum.

Einstein gravity has a family of solutions given by

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} \tag{41}$$

that is in accordance with the conservation laws for any small constant Λ. The vacuum solution of this equation with the Robertson–Walker metric with $k = 0$, gives us the de Sitter space–time:

$$ds^2 = dt^2 - \left[\exp 2 \left(\frac{1}{3}\Lambda \right)^{\frac{1}{2}} t \right] \left[dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right]. \tag{42}$$

If we add a cosmological constant in the vacuum equations of Visser gravity:

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T^{\text{mass}}_{\mu\nu}, \tag{43}$$

and taking the covariant divergence, from the right-hand side we reobtain Eq. (30). Since the background metric is Minkowski, from (14) the dynamical metric $g_{\mu\nu}$ is Minkowski too and we obtain

$$G_{\mu\nu} = T_{\mu\nu}^{\text{mass}} = 0, \tag{44}$$

and from (43) we have

$$\Lambda = 0. \tag{45}$$

Thus, in order to have consistency, Λ must be rigorously zero. Since the background metric needs to be Minkowski, the cosmological vacuum solution in Visser's theory is the static flat Minkowski space-time or, e.g., some kind of cosmological parameter (like $\Lambda(t)$). For this last alternative, we would have a coupling equation like (39), which would describe the evolution of the energy density of the vacuum component.

8 Conclusion

Our interpretation of the energy–momentum conservation in the Visser's massive gravity is in accordance to the equivalence principle and recover naturally the results of special relativity in the absence of gravitational sources.

The point of view considered in this paper allow us to consider Minkowski as background metric in Visser's theory in all astrophysical cases including cosmology.

This new interpretation may lead to interesting cosmological results once we can construct a cosmological model in a theory with massive gravitons with a Minkowski background. Additional contributions to the cosmological fluids will appear due to the modifications in the interaction potential, which, maybe, would be a way of treat the dark-energy problem. The analyses of the theory in the absence of gravitational sources lead us to exclude the de Sitter space-time as a vacuum solution of the massive gravity, once a constant Λ term is rigorously zero in a flat background.

Another interesting feature is that our interpretation of the energy conservation in strong fields is independent of the form of the tensor which interact with the perfect-fluid tensor, so this can be used to other models with additional energy–momentum contribution.

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